BPS Monopole in the Space of Boundary Conditions

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Based on:
SO, `arXiv:1506.04738[hep-th]`

Plan of the talk

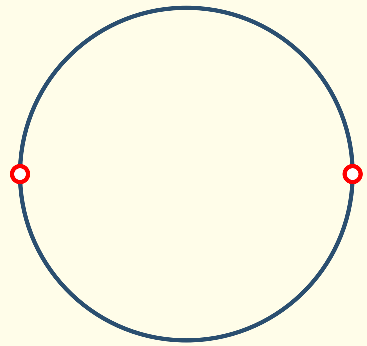
Overview

Today's talk: Quantum mechanics on a circle with two point-like interactions:

free Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

on



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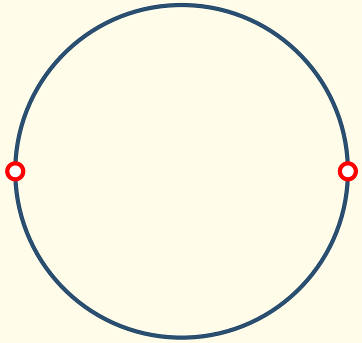

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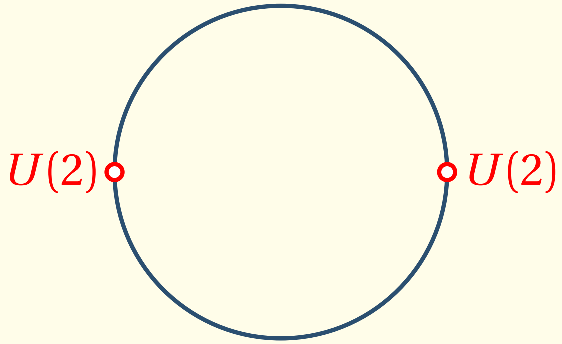

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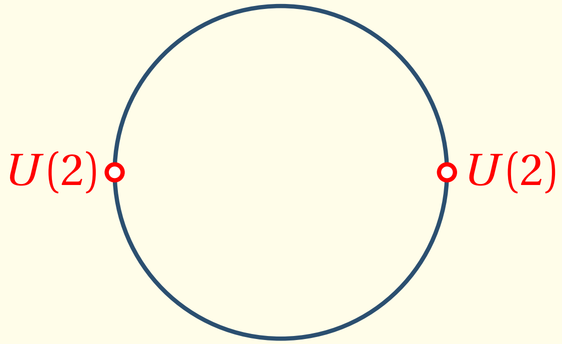

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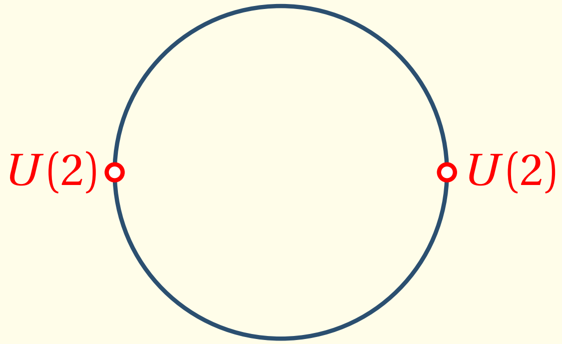


Interestingly enough, this simple model exhibits:

- a rich parameter space $U(2) \times U(2)$
- higher-derivative supersymmetry (in a subspace $\mathcal{M}_{\text{HSUSY}} \subset U(2) \times U(2)$)
- nontrivial non-Abelian Berry's phase $W_\gamma(A) = \mathcal{P}\exp\left(i\oint_\gamma A\right)$

path-ordered exponential

closed path $\gamma \subset \mathcal{M}_{\text{HSUSY}}$

Berry's connection (1-form on $\mathcal{M}_{\text{HSUSY}}$)

- **The goal of the talk** is to show that, in this model, Berry's connection $A^{(0)} = A_i^{(0)} dx_i$ in the ground-state sector is given by the celebrated Bogomolny-Prasad-Sommerfield (BPS) magnetic monopole

$$A_i^{(0)} = \epsilon_{ijk} \frac{x_j}{r^2} \frac{\sigma_k}{2} \left(1 - \frac{vr}{\sinh(vr)}\right) \quad (x_j\text{: local coordinates of } \mathcal{M}_{\mathrm{HSUSY}})$$

  which is one of the simplest classical solutions of four-dimensional $SU(2)$ Yang-Mills-Higgs theory

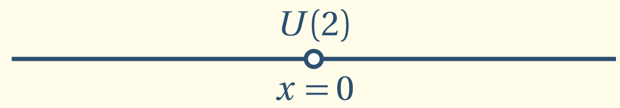
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} \left(\Phi^a \Phi^a - v^2\right)^2$$

- The keys to this result are $U(2)$ family of point-like interactions and higher-derivative supersymmetry.

- In the rest of the talk I will work in the units $\boldsymbol{\hbar = 2m = 1}$.

Point-like Interactions: A Review

- It has been long known that point-like interactions are all described by the $U(2)$ family of boundary conditions **[Fülöp-Tsutsui '99]**.



- The key to this is the **probability current conservation** at $x = 0$:

$$j(0_-) = j(0_+)$$

where $j(x)$ is the local probability current given by

$$j(x) = -i\left[\varphi^*(x)\varphi'(x) - \varphi^{*\prime}(x)\varphi(x)\right]$$

- The $U(2)$ family of boundary conditions is easy to derive! All you guys can follow it!

Here is the derivation: We require the probability current conservation at $x = 0$:

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$$\Leftrightarrow \begin{pmatrix} \varphi'(0_-) \\ -\varphi'(0_+) \end{pmatrix}^\dagger \begin{pmatrix} \varphi(0_-) \\ \varphi(0_+) \end{pmatrix} = \begin{pmatrix} \varphi(0_-) \\ \varphi(0_+) \end{pmatrix}^\dagger \begin{pmatrix} \varphi'(0_-) \\ -\varphi'(0_+) \end{pmatrix}$$

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$$\Longleftrightarrow \left| \begin{pmatrix} \varphi'(0_-) \\ -\varphi'(0_+) \end{pmatrix} - i\,v \begin{pmatrix} \varphi(0_-) \\ \varphi(0_+) \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \varphi'(0_-) \\ -\varphi'(0_+) \end{pmatrix} + i\,v \begin{pmatrix} \varphi(0_-) \\ \varphi(0_+) \end{pmatrix} \right|^2$$

(v: arbitrary reference scale of length dimension -1)

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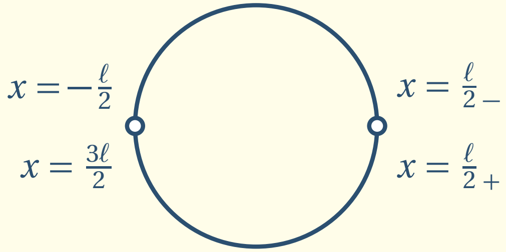
$$\Leftrightarrow \begin{pmatrix} \varphi'(0_-) \\ -\varphi'(0_+) \end{pmatrix} - i v \begin{pmatrix} \varphi(0_-) \\ \varphi(0_+) \end{pmatrix} = U \left[\begin{pmatrix} \varphi'(0_-) \\ -\varphi'(0_+) \end{pmatrix} + i v \begin{pmatrix} \varphi(0_-) \\ \varphi(0_+) \end{pmatrix} \right], \quad U \in U(2)$$

This describes the $U(2)$ family of point-like interactions in one dimension.

The same goes for point-like interactions on a circle. Hence a free spinless particle on S^1 with two point-like interactions



is described by the Schrödinger equation

$$-\varphi''(x) = E\varphi(x)$$

and the $U(2) \times U(2)$ family of boundary conditions

$$\begin{pmatrix} \varphi'(-\frac{\ell}{2}) \\ \varphi'(\frac{3\ell}{2}) \end{pmatrix} - iv \begin{pmatrix} \varphi(-\frac{\ell}{2}) \\ -\varphi(\frac{3\ell}{2}) \end{pmatrix} = U \left[\begin{pmatrix} \varphi'(-\frac{\ell}{2}) \\ \varphi'(\frac{3\ell}{2}) \end{pmatrix} + iv \begin{pmatrix} \varphi(-\frac{\ell}{2}) \\ -\varphi(\frac{3\ell}{2}) \end{pmatrix} \right], \quad U \in U(2)$$

$$\begin{pmatrix} \varphi'(\frac{\ell}{2}_-) \\ \varphi'(\frac{\ell}{2}_+) \end{pmatrix} - iv \begin{pmatrix} \varphi(\frac{\ell}{2}_-) \\ -\varphi(\frac{\ell}{2}_+) \end{pmatrix} = \bar{U} \left[\begin{pmatrix} \varphi'(\frac{\ell}{2}_-) \\ \varphi'(\frac{\ell}{2}_+) \end{pmatrix} + iv \begin{pmatrix} \varphi(\frac{\ell}{2}_-) \\ -\varphi(\frac{\ell}{2}_+) \end{pmatrix} \right], \quad \bar{U} \in U(2)$$

The Model

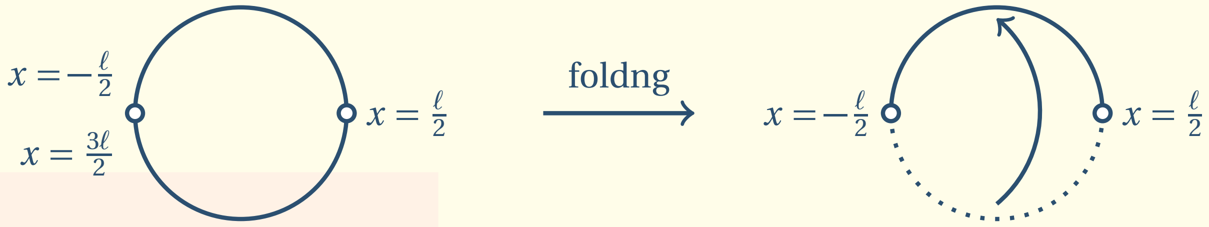
- Let $x \in (-\frac{\ell}{2}, \frac{3\ell}{2})$ be the coordinate of circle of circumference 2ℓ, where $x = -\frac{\ell}{2}$ and $x = \frac{3\ell}{2}$ are identified, and $\varphi(x)$ be the wavefunction on the circle. Point-like interactions are located at $x = \pm\frac{\ell}{2}$.
- **Folding trick**. It is customary to introduce a two-component vector-valued wavefunction $\boldsymbol{\psi}(x)$ on the interval $(-\frac{\ell}{2}, \frac{\ell}{2})$, whose upper- and lower-components are given by the wavefunctions on the upper- and lower-semicircles:

$$\boldsymbol{\psi}(x) = \begin{pmatrix} \varphi(x) \\ \varphi(\ell - x) \end{pmatrix}, \quad -\frac{\ell}{2} < x < \frac{\ell}{2}$$

- Mathematically speaking, we consider the following Hilbert space:

$$\mathcal{H} = L^2(-\tfrac{\ell}{2}, \tfrac{\ell}{2}) \oplus L^2(\tfrac{\ell}{2}, \tfrac{3\ell}{2}) \cong L^2(-\tfrac{\ell}{2}, \tfrac{\ell}{2}) \otimes \mathbb{C}^2$$

- Geometrically speaking, we fold the circle in half:



- In this way, quantum mechanics on the circle of circumference 2ℓ with scalar-valued wavefunctions is always mapped into quantum mechanics on the interval of length ℓ with vector-valued wavefunctions.
- Point-like interactions located at antipodal points of the circle are then translated into the problem of boundary conditions at the boundaries of the interval.
- Borrowing the terminology of boundary conformal field theory [Oshikawa-Affleck '96], we call the original scalar quantum mechanics the unfolding picture and the vector quantum mechanics the folding picture.
- The time-independent Schrödinger equation for a free particle in the folding picture is then given by the vector equation

$$-\boldsymbol{\psi}'' = E\boldsymbol{\psi}$$

- **Boundary conditions**. In general, two point-like interactions are described by the $U(2)\times U(2)$ family of boundary conditions. In this talk, however, we focus on the following $SU(2)$ subfamily of boundary conditions:

$$(1+U)\psi' - iv(1-U)\psi = 0 \quad \text{at} \quad x = \pm\frac{\ell}{2}$$

where $U \in SU(2) =: \mathcal{M}_{\text{HSUSY}}$.

- **Parameterization of U**. For the following discussions it is convenient to parameterize U into the form of spectral decomposition

$$U = \mathrm{e}^{i\alpha} P_+ + \mathrm{e}^{-i\alpha} P_-$$

where $\alpha \in [0,\pi]$ is an angle parameter. $P_\pm$ are projection operators and parameterized in two different ways:

$$P_\pm = \frac{1 \pm Z}{2} = \boldsymbol{e}_\pm^\dagger \boldsymbol{e}_\pm$$

where Z is a hermitian unitary matrix that satisfies $Z = Z^\dagger = Z^{-1}$ and $Z^2 = 1$ and $\boldsymbol{e}_\pm$ are orthonormal eigenvectors of Z (and U).

- Since the set of eigenvectors $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ provides the complete orthonormal basis, any element of $\mathcal{H} \cong L^2(-\frac{\ell}{2}, \frac{\ell}{2}) \otimes \mathbb{C}^2$ can be decomposed as follows:

$$\boldsymbol{\psi}(x) = \psi_+(x)\boldsymbol{e}_+ + \psi_-(x)\boldsymbol{e}_-$$

  We call the the set of eigenvectors $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ the basis and the coefficient functions $\{\psi_+, \psi_-\}$ the components.

- The boundary condition $(1+U)\boldsymbol{\psi}' - i\,v(1-U)\boldsymbol{\psi} = 0$ is now reduced to the following Robin boundary conditions for the components:

$$\begin{aligned} &\psi_+' - v(\alpha)\psi_+ = 0 \quad \text{and} \\ &\psi_-' + v(\alpha)\psi_- = 0 \quad \text{at} \quad x = \pm\frac{\ell}{2} \end{aligned}$$

  where $v(\alpha) = v\tan(\alpha/2)$.

- The Schrödinger equation boils down to the two independent differential equations for the components:

$$-\psi_\pm'' = E\psi_\pm$$

- **Doubly-degenerate ground states.** The lowest-energy eigenfunctions $\boldsymbol{\psi}_{\pm,0} = \psi_{\pm,0}\boldsymbol{e}_{\pm}$ are given by the zero-modes of differential operators $\frac{d}{dx} \pm v(\alpha)$ and take the following forms:

$$\boldsymbol{\psi}_{\pm,0}(x) = \sqrt{\frac{v(\alpha)}{\sinh(v(\alpha)\ell)}}\exp(\pm v(\alpha)x)\boldsymbol{e}_{\pm}$$

The ground-state energy is independent of ℓ and given by

$$E_0 = -v(\alpha)^2$$

- **Doubly-degenerate excited states.** Normalized energy eigenfunctions $\boldsymbol{\psi}_{\pm,n} = \psi_{\pm,n}\boldsymbol{e}_{\pm}$ turn out to be of the forms

$$\boldsymbol{\psi}_{\pm,n}(x) = \sqrt{\frac{2}{\ell}\frac{1}{1+\left(\frac{v(\alpha)\ell}{n\pi}\right)^2}}\left[\cos\left(\frac{n\pi}{\ell}\left(x+\frac{\ell}{2}\right)\right) \pm \frac{v(\alpha)\ell}{n\pi}\sin\left(\frac{n\pi}{\ell}\left(x+\frac{\ell}{2}\right)\right)\right]\boldsymbol{e}_{\pm}$$

The energy eigenvalues are independent of $v(\alpha)$ and given by

$$E_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, \cdots$$

- It would be reasonable to expect that there might be some underlying symmetry that ensures two-fold degeneracy.
- Below I shall show that the symmetry behind the doubly-degenerate energy levels is the so-called *second-order derivative supersymmetry*, which is a nonlinear extension of $\mathcal{N}=2$ quantum mechanical supersymmetry introduced by Andrianov *et al.* [Andrianov-Ioffe-Spiridonov '93]
- Just as in the case of $\mathcal{N}=2$ quantum mechanical supersymmetry, the second-order derivative supersymmetry algebra consists of four operators:
  - **Hamiltonian $\boldsymbol{H}$** (second-order derivative operator)
  - **supercharge $\boldsymbol{Q^+}$ & its adjoint $\boldsymbol{Q^-}$** (second-order derivative operators)
  - **fermion parity $\boldsymbol{(-1)^F}$** ($\mathbb{Z}_2$ grading operator)

- The Robin boundary conditions

$$-\psi'_+ + v(\alpha)\psi_+ = 0$$
$$+\psi'_- + v(\alpha)\psi_- = 0$$

motivate us to introduce the following first-order differential operators:

$$A^{\pm} = \pm\frac{d}{dx} + v(\alpha)$$

- It is easy to check that the components $\psi_{\pm,n}$ satisfy the following relations:

$$A^{\mp}A^{\mp}\psi_{\pm,n} = \pm(E_n - E_0)\psi_{\mp,n}$$

- Now we are in a position to introduce the second-order derivative supersymmetry algebra. In the basis in which U becomes diagonal, the set of operators $\{H, Q^{\pm}, (-1)^F\}$ is given by

$$H = \begin{pmatrix} A^+A^- + E_0 & 0 \\ 0 & A^-A^+ + E_0 \end{pmatrix}$$
$$Q^+ = \begin{pmatrix} 0 & 0 \\ A^-A^- & 0 \end{pmatrix}$$
$$Q^- = \begin{pmatrix} 0 & A^+A^+ \\ 0 & 0 \end{pmatrix}$$
$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which act on the energy eigenfunctions as follows:

$$H\boldsymbol{\psi}_{\pm,n} = E_n\boldsymbol{\psi}_{\pm,n}$$
$$Q^{\pm}\boldsymbol{\psi}_{\pm,n} = \pm(E_n - E_0)\boldsymbol{\psi}_{\mp,n}$$
$$(-1)^F\boldsymbol{\psi}_{\pm,n} = \pm\boldsymbol{\psi}_{\pm,n}$$

- The set of operators $\{H, Q^{\pm}, (-1)^F\}$ satisfies the following relations of second-order derivative supersymmetry algebra:

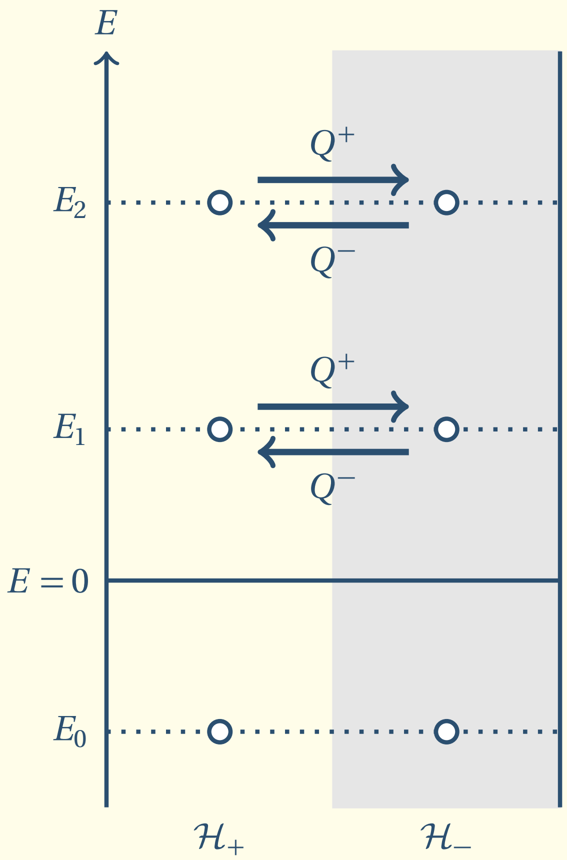
$$
\begin{aligned}
&(Q^{\pm})^2 = 0 \\
&\big((-1)^F\big)^2 = 1 \\
&[H, Q^{\pm}] = [H, (-1)^F] = 0 \\
&\{Q^{\pm}, (-1)^F\} = 0 \\
&\{Q^{+}, Q^{-}\} = (H - E_0)^2
\end{aligned}
$$

- Thanks to the fermion parity $(-1)^F$ the Hilbert space splits into two orthogonal subspaces

$$\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$$

where $\mathcal{H}_{\pm} = \{\psi \in \mathcal{H} : (-1)^F \psi = \pm\psi\}$ are “bosonic” and “fermionic” subspaces, respectively.

- Here is the schematic structure of doubly-degenerate energy levels. Arrows indicate the second-order derivative supersymmetry transformations between “bosonic” and “fermionic” states.



Non-Abelian Berry's Phase

- Let us now move on to the analysis of time-dependent situation where boundary condition parameters vary very slowly.
- Suppose that the initial state ψ_{in} at time $t = 0$ is in the subspace of nth excited state, $\psi_{\text{in}} \in \mathcal{H}_n = \text{span}\{\psi_{+,n}, \psi_{-,n}\}$.
- At time $t = T$, transitions between different subspaces $\mathcal{H}_n$ and $\mathcal{H}_m$ $(n \neq m)$ are suppressed by the factor $1/T$ such that the initial state $\psi_{\text{in}} \in \mathcal{H}_n$ remains in the subspace $\mathcal{H}_n$ in the adiabatic limit $T \to \infty$.
- Under an adiabatic time-evolution along a closed path γ on the parameter space $\mathcal{M}_{\text{HSUSY}} = SU(2)$, the initial state ψ_{in} transforms into the final state

$$\psi_{\text{out}} = \mathrm{e}^{-i\int_0^T dt\, E_n} W_\gamma(A^{(n)})\psi_{\text{in}} \in \mathcal{H}_n$$

where $\mathrm{e}^{-i\int_0^T dt\, E_n}$ is the T-dependent dynamical phase and $W_\gamma(A^{(n)})$ is the T-independent geometric phase given by the Wilson loop **[Wilczek-Zee '84]**

$$W_\gamma(A^{(n)}) = \mathcal{P}\exp\left(i\oint_\gamma A^{(n)}\right)$$

- $A^{(n)} = (A^{(n)}_{ab})$ is the 2×2 hermitian matrix-valued one-form, or Berry's connection, for the nth excited sector and given by the inner product

$$A^{(n)}_{ab} = i\langle \psi_{a,n} | d | \psi_{b,n} \rangle = i \int_{-\ell/2}^{\ell/2} dx\, \psi^{\dagger}_{a,n}(x) d\psi_{b,n}(x), \quad a, b \in \{+, -\}$$

  where d stands for the exterior derivative on the space of boundary conditions $\mathcal{M}_{\text{HSUSY}} = SU(2) \cong S^3$.

- Note that under the unitary change of the basis

$$\psi_{a,n} \mapsto \tilde{\psi}_{a,n} = \psi_{b,n} g_{ba}, \quad g = (g_{ba}) \in SU(2)$$

  the Berry connection indeed transforms as a connection

$$A^{(n)} \mapsto \tilde{A}^{(n)} = g^{\dagger} A^{(n)} g + i g^{\dagger} dg$$

- I shall show that $A^{(n)}$ is given by the BPS 't Hooft-Polyakov monopole for $n = 0$ and non-BPS 't Hooft-Polyakov monopole for $n \geq 1$.

- **"String" gauge**. Substituting the normalized energy eigenfunctions we get

$$A^{(n)} = \begin{pmatrix} i\boldsymbol{e}_+^\dagger d\boldsymbol{e}_+ & iK^{(n)}\boldsymbol{e}_+^\dagger d\boldsymbol{e}_- \\ iK^{(n)}\boldsymbol{e}_-^\dagger d\boldsymbol{e}_+ & i\boldsymbol{e}_-^\dagger d\boldsymbol{e}_- \end{pmatrix}$$

where $K^{(n)}$ is given by the overlapping integral

$$K^{(n)} = \int_{-\ell/2}^{\ell/2} dx\, \psi_{\pm,n}^*(x)\psi_{\mp,n}(x) = \begin{cases} \dfrac{v(\alpha)\ell}{\sinh(v(\alpha)\ell)} & \text{for } n=0 \\ \dfrac{1-(v(\alpha)\ell/n\pi)^2}{1+(v(\alpha)\ell/n\pi)^2} & \text{for } n\geq 1 \end{cases}$$

- The hermitian unitary matrix Z is parameterized as $Z = \boldsymbol{n}\cdot\boldsymbol{\sigma}$, where $\boldsymbol{n}$ is a unit 3-vector. In the spherical coordinates $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the orthonormal eigenvectors $\boldsymbol{e}_\pm$ of $Z = \boldsymbol{n}\cdot\boldsymbol{\sigma}$ are chosen to be of the forms

$$\boldsymbol{e}_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{e}_- = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

where $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$ are polar and azimuthal angles of S^2.

- In the "string" gauge Berry's connection turns out to be of the form $A^{(n)} = A^{(n)}_\theta d\theta + A^{(n)}_\phi d\phi$, where

$$A^{(n)}_\theta = -K^{(n)} \sin\phi \frac{\sigma_1}{2} + K^{(n)} \cos\phi \frac{\sigma_2}{2}$$
$$A^{(n)}_\phi = -K^{(n)} \sin\theta \cos\phi \frac{\sigma_1}{2} - K^{(n)} \sin\theta \sin\phi \frac{\sigma_2}{2} - (1-\cos\theta)\frac{\sigma_3}{2}$$

- **Dirac string**. Note that $A^{(n)}_\phi$ is ill-defined at the south pole $\theta = \pi$ because the combination $1-\cos\theta$ does not vanish at $\theta = \pi$. In other words, the Berry connection suffers from the Dirac string singularity along the negative 3-axis.

- In fact, the eigenvectors $\boldsymbol{e}_\pm$ themselves are not globally well-defined over the whole 2-sphere and suffer from the Dirac string. (Notice that the combinations $\mathrm{e}^{\pm i\phi} \sin\frac{\theta}{2}$ do not vanish at $\theta = \pi$.)

- This Dirac string singularity, however, can be removed by *singular gauge transformation* [Arafune-Freund-Goebel '75]. Below I shall perform this and transform $A^{(n)}$ into a manifestly spherically symmetric form that is more familiar in gauge theory.

- **"Hedgehog" gauge.** Let us next move into the following gauge:

$$g = \begin{pmatrix} \boldsymbol{e}_+^\dagger \\ \boldsymbol{e}_-^\dagger \end{pmatrix}$$

  Note that this unitary matrix inherits the Dirac string singularity from the eigenvectors $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ and hence is not globally well-defined over the whole parameter space.

- A straightforward calculation gives

$$\begin{aligned} \tilde{A}^{(n)} &= \begin{pmatrix} \boldsymbol{e}_+ & \boldsymbol{e}_- \end{pmatrix} \begin{pmatrix} i\,\boldsymbol{e}_+^\dagger d\,\boldsymbol{e}_+ & i\,K^{(n)}\boldsymbol{e}_+^\dagger d\,\boldsymbol{e}_- \\ i\,K^{(n)}\boldsymbol{e}_-^\dagger d\,\boldsymbol{e}_+ & i\,\boldsymbol{e}_-^\dagger d\,\boldsymbol{e}_- \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_+^\dagger \\ \boldsymbol{e}_-^\dagger \end{pmatrix} + i \begin{pmatrix} \boldsymbol{e}_+ & \boldsymbol{e}_- \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_+^\dagger \\ \boldsymbol{e}_-^\dagger \end{pmatrix} \\ &= \frac{i}{2}\left(1 - K^{(n)}\right) Z\,d\,Z \end{aligned}$$

  where $Z = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{e}_+\boldsymbol{e}_+^\dagger - \boldsymbol{e}_-\boldsymbol{e}_-^\dagger$ is a hermitian unitary matrix.

- Note that $\tilde{A}^{(n)}$ vanishes for $K^{(n)} = 1$ and becomes pure gauge $i\,Z\,d\,Z$ for $K^{(n)} = -1$.

- Let us parameterize the unit 3-vector $\boldsymbol{n}$ into the following "hedgehog" configuration:

$$\boldsymbol{n} = \frac{\boldsymbol{r}}{r}$$

where $\boldsymbol{r} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq 0$. With this parameterization the one-form $iZdZ$ becomes

$$\begin{aligned} iZdZ &= i\frac{x_i\sigma_i}{r}\frac{\partial}{\partial x_k}\left(\frac{x_j\sigma_j}{r}\right)dx_k \\ &= \epsilon_{ijk}\frac{x_j\sigma_k}{r^2}dx_i \end{aligned}$$

- Though r can be arbitrary from the viewpoint of parameterization of $\boldsymbol{n}$, below I shall fix the length r to satisfy the following relation:

$$r = \ell\tan\left(\frac{\alpha}{2}\right), \quad \alpha \in [0, \pi]$$

- Under the identification $r = \ell \tan(\frac{\alpha}{2})$, the Berry connection $\tilde{A}^{(n)} = \tilde{A}_i^{(n)} dx_i$ takes the form of 't Hooft-Polyakov ansatz ['t Hooft '74] [Polyakov '74]

$$\tilde{A}_i^{(n)} = \epsilon_{ijk} \frac{x_j}{r^2} \frac{\sigma_k}{2} \left(1 - K^{(n)}(r)\right)$$

where

$$K^{(n)}(r) = \begin{cases} \dfrac{v r}{\sinh(v r)} & \text{for } n = 0 \\[2ex] \dfrac{1 - (v r / n\pi)^2}{1 + (v r / n\pi)^2} & \text{for } n \geq 1 \end{cases}$$

- Note that $K^{(n)}$ has the following asymptotic behaviors:

$$K^{(n)}(r) \overset{r \to 0}{\longrightarrow} 1 + O(r^2) \quad \text{and} \quad K^{(n)}(r) \overset{r \to \infty}{\longrightarrow} -1 + O(r^{-2})$$

Hence $\tilde{A}^{(n)} = \frac{i}{2}(1 - K^{(n)}) Z dZ$ vanishes at $r = 0$ and becomes pure gauge $iZdZ$ at $r = \infty$, which is the desired properties of 't Hooft-Polyakov monopole. Note also that $\tilde{A}_i^{(0)}$ is nothing but the BPS monopole solution of $SU(2)$ Yang-Mills-Higgs theory [Prasad-Sommerfield '75].

- In 2008, Sonner and Tong constructed a quantum mechanical model for a spin-1/2 on S^2 where Berry's connection and the matrix elements of the operator $\cos\hat{\theta}$ become the BPS solutions for gauge and Higgs fields of $SU(2)$ Yang-Mills-Higgs theory [Sonner-Tong '08].
- Motivated by their results, in the rest of the talk I would like to construct a quantum mechanical counterpart of Higgs field in our free particle model.
- Let us consider the following matrix elements of position operator $\hat{x}$ in the ground-state sector:

$$\Phi^{(0)}_{ab} = \frac{v}{\ell}\langle\psi_{a,0}|\hat{x}|\psi_{b,0}\rangle = \frac{v}{\ell}\int_{-\ell/2}^{\ell/2} dx\, \psi^{\dagger}_{a,0}(x) x \psi_{b,0}(x)$$

- Note that the gauge transformation

$$\psi_{a,0} \mapsto \tilde{\psi}_{a,0} = \psi_{b,0} g_{ba}, \quad g = (g_{ba}) \in SU(2)$$

acts on the 2×2 hermitian matrix $\Phi^{(0)} = (\Phi^{(0)}_{ab})$ as the adjoint action

$$\Phi^{(0)} \mapsto \tilde{\Phi}^{(0)} = g^{\dagger}\Phi^{(0)} g$$

- In the "string" gauge, $\Phi^{(0)}$ takes the following diagonal form:

$$\Phi^{(0)} = \begin{pmatrix} vH^{(0)} & 0 \\ 0 & -vH^{(0)} \end{pmatrix}$$

where

$$H^{(0)} = \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} dx\, \psi^*_{+,0}(x)\, x\, \psi_{+,0}(x) = \frac{1}{2}\left(\coth(v(\alpha)\ell) - \frac{1}{v(\alpha)\ell}\right)$$

- In the "hedgehog" gauge $g = \begin{pmatrix} \boldsymbol{e}_+^\dagger \\ \boldsymbol{e}_-^\dagger \end{pmatrix}$, $\tilde{\Phi}^{(0)}$ takes the following form:

$$\begin{aligned}\tilde{\Phi}^{(0)} &= \begin{pmatrix} \boldsymbol{e}_+ & \boldsymbol{e}_- \end{pmatrix} \begin{pmatrix} vH^{(0)} & 0 \\ 0 & -vH^{(0)} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_+^\dagger \\ \boldsymbol{e}_-^\dagger \end{pmatrix} = vH^{(0)} Z \\ &= v\frac{x_i}{r}\frac{\sigma_i}{2}\left(\coth(vr) - \frac{1}{vr}\right)\end{aligned}$$

which is exactly the same form as the BPS solution for the Higgs field in $SU(2)$ Yang-Mills-Higgs theory **[Prasad-Sommerfield '75]**.

Summary

- I have constructed a simple quantum mechanical model for a free spinless particle in which all the energy levels become doubly-degenerate thanks to the hidden higher-derivative supersymmetry.
- I then showed that, in the ground-state/excited-state sector of the model, Berry's connections are given by BPS/non-BPS 't Hooft-Polyakov monopoles of $SU(2)$ Yang-Mills-Higgs theory.
- I also showed that the matrix elements of position operator in the ground-state sector gives the BPS solution of adjoint Higgs field in $SU(2)$ Yang-Mills-Higgs theory.

	ground-state sector	excited-state sector
Berry's connection	BPS monopole	non-BPS monopole
matrix elements of $\hat{x}$	adjoint Higgs field	N/A

Thank you for your attention!

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