

Recurrence Relations for Finite-Temperature Correlators via $\text{AdS}_2/\text{CFT}_1$

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Introduction and Overview

- In d -dimensional Minkowski spacetime, $SO(2, d)$ global conformal transformations consist of the followings:

$$\text{(translation)} \quad x^\mu \mapsto x'^\mu = x^\mu + a^\mu$$

$$\text{(Lorentz transformation)} \quad x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\text{(dilatation)} \quad x^\mu \mapsto x'^\mu = \lambda x^\mu$$

$$\text{(conformal boost)} \quad x^\mu \mapsto x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b x + b^2 x^2}$$

- Basic elements of conformal field theory (CFT) are (quasi-)primary operators $\mathcal{O}_\Delta(x)$, which transform under the $SO(2, d)$ global conformal transformations $x^\mu \mapsto x'^\mu$ as follows:

$$\mathcal{O}_\Delta(x) \mapsto \mathcal{O}'_\Delta(x) = \left| \frac{\partial x'}{\partial x} \right|^{\Delta/d} \mathcal{O}_\Delta(x')$$

where $|\partial x' / \partial x|$ stands for the Jacobian of the global conformal transformation. Δ is called the conformal weight of $\mathcal{O}_\Delta(x)$.

- Let $|0\rangle$ be the conformally invariant vacuum. Then the n -point function for the (quasi-)primary operators satisfies the identity

$$\langle 0 | \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) | 0 \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\Delta_n/d} \langle 0 | \mathcal{O}_{\Delta_1}(x'_1) \cdots \mathcal{O}_{\Delta_n}(x'_n) | 0 \rangle$$

- It has been long known that the $SO(2, d)$ global conformal symmetry completely fixes the possible forms of two- and three-point functions in any spacetime dimension d [Polyakov '70]. Indeed, up to overall normalization factors, they can be determined as follows:

$$\langle 0 | \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) | 0 \rangle = \delta_{\Delta_1 \Delta_2} \frac{C_{\Delta_1 \Delta_2}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}$$

$$\langle 0 | \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) | 0 \rangle = \frac{C_{\Delta_1 \Delta_2 \Delta_3}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1} |x_3 - x_1|^{\Delta_3 + \Delta_1 - \Delta_2}}$$

- Conformal constraints work well in coordinate space; however, they tell us little about momentum-space correlators before performing Fourier transform.
- Indeed, in spite of its simplicity in coordinate space, three-point functions in momentum space are known to be very complicated.
 - zero-temperature CFT_d : [Corianò-Delle Rose-Mottola-Serino '13] [Bzowski-McFadden-Skenderis '13]
 - finite-temperature CFT_2 : [Becker-Cabrera-Su '14]
- It is desirable to understand how conformal symmetry restricts the possible forms of momentum-space correlators. Here are the reasons:
 - ◆ Momentum-space correlators are directly related to physical observables (such as spectral density);
 - ◆ Fourier transform of position-space correlators is very hard.
- Today I am going to present a small example that, by using the AdS/CFT correspondence, finite-temperature CFT_1 two-point functions in frequency space can also be determined by conformal symmetry.

- **Scope of the talk:** Klein-Gordon equation on charged AdS₂ black hole

$$(\square_{\text{AdS}_2} - m^2)\Phi(T, X) = 0$$

and the asymptotic near-boundary behavior of general solution

$$\Phi(T, X) \sim A_\Delta(\omega)X^\Delta e^{-i\omega T} + B_\Delta(\omega)X^{1-\Delta}e^{-i\omega T} \quad \text{as } X \rightarrow 0$$

- According to the real-time prescription of AdS/CFT correspondence, the ratio

$$G_\Delta(\omega) = (2\Delta - 1) \frac{A_\Delta(\omega)}{B_\Delta(\omega)}$$

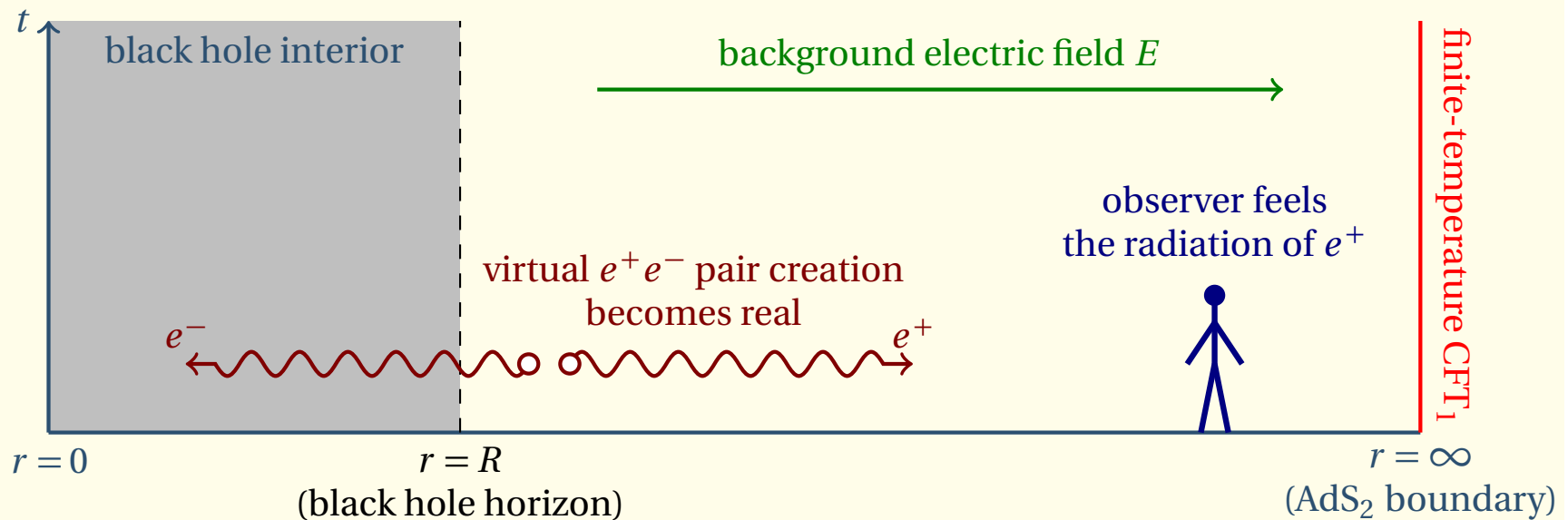
gives the frequency-space two-point function for a charged scalar operator of conformal weight Δ in dual finite-temperature CFT₁ [Iqbal-Liu '09].

- **My small findings:** The ratio $A_\Delta(\omega)/B_\Delta(\omega)$ can be computed without solving the Klein-Gordon equation: it is determined only through symmetry of the AdS₂ black hole.

AdS₂ in a Nutshell

- Charged AdS₂ black hole is a *locally* AdS₂ spacetime and described by the following metric and gauge field:

$$ds_{\text{AdS}_2}^2 = -\left(\left(\frac{r}{R}\right)^2 - 1\right) dt^2 + \frac{dr^2}{(r/R)^2 - 1} \quad \text{and} \quad A = -E(r - R)dt$$



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- It is known that any black holes have an AdS₂ factor in the near-horizon limit [**AdS₂ structure theorem** [Kunduri-Lucietti-Reall '07]]. In this sense the AdS₂ black hole plays a special role in black hole physics and holography.
- For the following discussions it is convenient to introduce a new coordinate system (t, x) via

$$r = R \coth(x/R), \quad x \in (0, \infty)$$

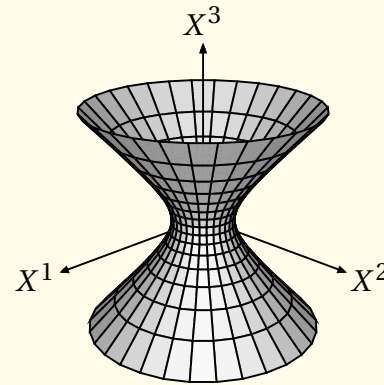
in which the metric and gauge field become

$$ds_{\text{AdS}_2}^2 = \frac{-dt^2 + dx^2}{\sinh^2(x/R)} \quad \text{and} \quad A = -ER(\coth(x/R) - 1)dt$$

- **Below I will work in the units $R = 1$.**

- AdS₂ spacetime is a 2d spacetime with negative constant curvature -1 . It is embedded into the ambient space $\mathbb{R}^{2,1}$ and defined as the 2d hyperboloid

$$\text{AdS}_2 = \{(X^1, X^2, X^3) \in \mathbb{R}^{2,1} : -(X^1)^2 - (X^2)^2 + (X^3)^2 = -1\}$$



- AdS₂ metric is then given by the following induced metric

$$ds_{\text{AdS}_2}^2 = -(dX^1)^2 - (dX^2)^2 + (dX^3)^2 \Big|_{-(X^1)^2 - (X^2)^2 + (X^3)^2 = -1}$$

- The hyperboloid is obviously invariant under the $SO(2, 1)$ transformations

$$X^\mu \mapsto X'^\mu = \Lambda^\mu{}_\nu X^\nu, \quad \Lambda \in SO(2, 1) \cong SL(2, \mathbb{R})/\mathbb{Z}_2$$

isometry group of AdS₂ = $SO(2, 1) (\cong SL(2, \mathbb{R})/\mathbb{Z}_2)$

- The Lie algebra $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$ of the Lie group $SO(2, 1) \cong SL(2, \mathbb{R})/\mathbb{Z}_2$ is spanned by the following traceless matrices:

$$iJ_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad iJ_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iJ_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which satisfy the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = -iJ_2$$

- These generators generate the following one-parameter subgroups:

$$\exp(i\epsilon J_1) = \begin{pmatrix} \cosh \frac{\epsilon}{2} & \sinh \frac{\epsilon}{2} \\ \sinh \frac{\epsilon}{2} & \cosh \frac{\epsilon}{2} \end{pmatrix} \in SO(1, 1)$$

$$\exp(i\epsilon J_2) = \begin{pmatrix} e^{\epsilon/2} & 0 \\ 0 & e^{-\epsilon/2} \end{pmatrix} \in SO(1, 1)$$

$$\exp(i\epsilon J_3) = \begin{pmatrix} \cos \frac{\epsilon}{2} & \sin \frac{\epsilon}{2} \\ -\sin \frac{\epsilon}{2} & \cos \frac{\epsilon}{2} \end{pmatrix} \in SO(2)$$

- The linear combinations

$$iH = i(J_1 + J_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad iD = iJ_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iK = i(J_1 - J_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfy the (0 + 1)-dimensional conformal algebra

$$[H, D] = iH, \quad [D, K] = iK, \quad [K, H] = 2iD$$

- These generators generate the following one-parameter subgroups:

$$\exp(i\epsilon H) = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \in E(1)$$

$$\exp(i\epsilon D) = \begin{pmatrix} e^{\epsilon/2} & 0 \\ 0 & e^{-\epsilon/2} \end{pmatrix} \in SO(1, 1)$$

$$\exp(i\epsilon K) = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} \in E(1)$$

- It is known that the Lie group $SO(2, 1)$ contains only three distinct one-parameter subgroups: compact rotation group $SO(2)$, noncompact Euclidean group $E(1)$ and noncompact Lorentz group $SO(1, 1)$.
- Correspondingly, there exist three distinct classes of coordinates patches on AdS_2 where time-translation generators generate the one-parameter subgroups $SO(2)$, $E(1)$ and $SO(1, 1)$.
- I shall show that the time-translation generator of AdS_2 black hole generates the one-parameter subgroup $SO(1, 1) \subset SO(2, 1)$, which, after Wick rotation, becomes the compact rotation group $SO(2)$ and hence leads to quantized Matsubara frequencies conjugate to the imaginary time.

Here are the three distinct coordinate patches on the 2d hyperboloid
 $-(X^1)^2 - (X^2)^2 + (X^3)^2 = -1$:

1. **Global coordinates:** $SO(2)$ diagonal basis¹

$$(X^1, X^2, X^3) = \left(\frac{\sin \tau}{\sin \sigma}, \frac{\cos \tau}{\sin \sigma}, \frac{\cos \sigma}{\sin \sigma} \right), \quad ds_{\text{global}}^2 = \frac{-d\tau^2 + d\sigma^2}{\sin^2 \sigma}$$

2. **Poincaré coordinates:** $E(1)$ diagonal basis

$$(X^1, X^2, X^3) = \left(\frac{t}{x}, \frac{1-t^2+x^2}{2x}, \frac{1+t^2-x^2}{2x} \right), \quad ds_{\text{Poincaré}}^2 = \frac{-dt^2 + dx^2}{x^2}$$

3. **Schwarzschild coordinates:** $SO(1,1)$ diagonal basis

$$(X^1, X^2, X^3) = \left(\frac{\sinh T}{\sinh X}, \frac{\cosh X}{\sinh X}, \frac{\cosh T}{\sinh X} \right), \quad ds_{\text{Schwarzschild}}^2 = \frac{-dT^2 + dX^2}{\sinh^2 X}$$

¹ τ ranges from $-\infty$ to $+\infty$ for the covering space of AdS_2 without identification $\tau \sim \tau + 2\pi$.

1. **Global coordinates.** The line element

$$ds_{\text{global}}^2 = \frac{-d\tau^2 + d\sigma^2}{\sin^2 \sigma}$$

is invariant under the following $SL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 1)$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \sigma^\pm \mapsto \sigma'^\pm = 2 \arctan \left(\frac{a \tan \frac{\sigma^\pm}{2} + b}{c \tan \frac{\sigma^\pm}{2} + d} \right), \quad \sigma^\pm := \tau \pm \sigma$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ and $(a, b, c, d) \sim (-a, -b, -c, -d)$. It is easy to check that the $SO(2)$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \frac{\epsilon}{2} & \sin \frac{\epsilon}{2} \\ -\sin \frac{\epsilon}{2} & \cos \frac{\epsilon}{2} \end{pmatrix} \in SO(2)$$

induces the translation $\sigma'^\pm = \sigma^\pm + \epsilon$, which in the original coordinates reads

$$\tau' = \tau + \epsilon, \quad \sigma' = \sigma$$

time translation = SO(2) transformation

2. **Poincaré coordinates.** The line element

$$ds_{\text{Poincaré}}^2 = \frac{-dt^2 + dx^2}{x^2}$$

is invariant under the following $SL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 1)$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: x^\pm \mapsto x'^\pm = \frac{ax^\pm + b}{cx^\pm + d}, \quad x^\pm := t \pm x$$

It is easy to check that the $E(1)$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \in E(1)$$

induces the translation $x'^\pm = x^\pm + \epsilon$, which in the original coordinates reads

$$t' = t + \epsilon, \quad x' = x$$

time translation = $E(1)$ transformation

3. **Schwarzschild coordinates.** The line element

$$ds_{\text{Schwarzschild}}^2 = \frac{-dT^2 + dX^2}{\sinh^2 X}$$

is invariant under the following $SL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 1)$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: X^\pm \mapsto X'^\pm = 2 \operatorname{arctanh} \left(\frac{a \tanh \frac{X^\pm}{2} + b}{c \tanh \frac{X^\pm}{2} + d} \right), \quad X^\pm := T \pm X$$

It is easy to check that the $SO(1, 1)$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cosh \frac{\epsilon}{2} & \sinh \frac{\epsilon}{2} \\ \sinh \frac{\epsilon}{2} & \cosh \frac{\epsilon}{2} \end{pmatrix} \in SO(1, 1)$$

induces the translation $X'^\pm = X^\pm + \epsilon$, which in the original coordinates reads

$$T' = T + \epsilon, \quad X' = X$$

time translation = $SO(1, 1)$ transformation

- Let us consider infinitesimal forms of $SO(2, 1)$ coordinate transformations

$$\exp(i\epsilon J_a) : x^\mu \mapsto x'^\mu = x^\mu + \delta_a x^\mu + O(\epsilon^2)$$

- Background gauge field.** Let A_μ be a background gauge field that satisfies $dA = E \sqrt{|g|} dx^0 \wedge dx^1$, where E is a constant background electric field. Under the $SO(2, 1)$ coordinate transformations, A_μ transforms as a vector:

$$\exp(i\epsilon J_a) : A_\mu(x) \mapsto A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

Interestingly, the infinitesimal field variations $\delta_a A_\mu$ turn out to be of the forms of *gauge transformations*:

$$\begin{aligned} \delta_a A_\mu(x) &= A'_\mu(x) - A_\mu(x) \\ &= -\delta_a x^\nu \partial_\nu A_\mu(x) - A_\nu(x) \partial_\mu \delta_a x^\nu + O(\epsilon^2) \\ &= \partial_\mu \Lambda_a(x) + O(\epsilon^2) \end{aligned}$$

where Λ_a are some scalar functions.

- **Scalar field.** Let Φ be a charged scalar field that couples to the background gauge field via the covariant derivative $D_\mu = \partial_\mu - iqA_\mu$. Since the $SO(2, 1)$ coordinate transformations act on A_μ as gauge transformations, transformation law of Φ must be accompanied with the $U(1)$ gauge transformation. Hence

$$\exp(i\epsilon J_a) : \Phi(x) \mapsto \Phi'(x') = e^{iq\Lambda_a(x)}\Phi(x)$$

from which we find

$$\begin{aligned} \delta_a \Phi(x) &= \Phi'(x) - \Phi(x) \\ &= -\delta_a x^\mu \partial_\mu \Phi(x) + iq\Lambda_a(x)\Phi(x) + O(\epsilon^2) \\ &= i\epsilon J_a \Phi(x) + O(\epsilon^2) \end{aligned}$$

where J_a are the coordinate representations of $SO(2, 1)$ generators given by

$$i\epsilon J_a = -\delta_a x^\mu \partial_\mu + iq\Lambda_a(x)$$

1. **Global coordinates.** The $SO(2, 1)$ generators are turned out to be of the forms

$$J_3 = i\partial_\tau$$

$$J_\pm = -J_1 \pm iJ_2 = e^{\mp i\tau} \sin\sigma [\mp\partial_\sigma - \cot\sigma(i\partial_\tau) - \alpha]$$

where $\alpha := qE = qER^2$. The quadratic Casimir $C = -J_1^2 - J_2^2 + J_3^2$ is given by

$$C = \sin^2\sigma [-\partial_\tau^2 + \partial_\sigma^2 - \alpha^2 - 2\alpha \cot\sigma(i\partial_\tau)]$$

We are interested in the basis in which the time-translation generator becomes diagonal. Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of J_3 and C that satisfies the eigenvalue equations

$$J_3|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle \quad \text{and} \quad C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle$$

which in the coordinate space become the following differential equations:

$$i\partial_\tau\Phi_{\Delta,\omega}(\tau, \sigma) = \omega\Phi_{\Delta,\omega}(\tau, \sigma)$$

$$\left[-\partial_\sigma^2 + \frac{\Delta(\Delta - 1)}{\sin^2\sigma} + 2\omega\alpha \cot\sigma \right] \Phi_{\Delta,\omega}(\tau, \sigma) = (\omega^2 - \alpha^2)\Phi_{\Delta,\omega}(\tau, \sigma)$$

2. **Poincaré coordinates.** The $SO(2, 1)$ generators are given by

$$H = J_1 + J_3 = i\partial_t$$

$$D = J_2 = i(t\partial_t + x\partial_x)$$

$$K = J_1 - J_3 = -i(t^2 + x^2)\partial_t - 2itx\partial_x + 2\alpha x$$

The quadratic Casimir $C = -\frac{1}{2}(HK + KH) - D^2$ is

$$C = x^2(-\partial_t^2 + \partial_x^2) - 2i\alpha x\partial_t$$

Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of H and C that satisfies the eigenvalue equations

$$H|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle \quad \text{and} \quad C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle$$

which in the coordinate space become the following differential equations:

$$i\partial_t\Phi_{\Delta,\omega}(t, x) = \omega\Phi_{\Delta,\omega}(t, x)$$

$$\left[-\partial_x^2 + \frac{\Delta(\Delta - 1)}{x^2} + \frac{2\omega\alpha}{x} \right] \Phi_{\Delta,\omega}(t, x) = \omega^2\Phi_{\Delta,\omega}(t, x)$$

3. **Schwarzschild coordinates.** The $SO(2, 1)$ generators are turned out to be of the forms

$$L_1 = J_1 = i\partial_T + \alpha$$

$$L_{\pm} = J_2 \pm J_3 = \pm e^{\pm T} \sinh X [\pm i\partial_X + \coth X (i\partial_T + \alpha) - \alpha]$$

The quadratic Casimir $C = -L_1(L_1 \pm i) - L_{\mp}L_{\pm}$ is given by

$$C = \sinh^2 X [(i\partial_T + \alpha)^2 + \partial_X^2 + \alpha^2 - 2\alpha \coth X (i\partial_T + \alpha)]$$

Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of L_1 and C that satisfies the eigenvalue equations

$$L_1|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle \quad \text{and} \quad C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle$$

which in the coordinate space become the following differential equations:

$$i\partial_T \Phi_{\Delta, \omega}(T, X) = (\omega - \alpha)\Phi_{\Delta, \omega}(T, X)$$

$$\left[-\partial_X^2 + \frac{\Delta(\Delta - 1)}{\sinh^2 X} + 2\omega\alpha \coth X \right] \Phi_{\Delta, \omega}(T, X) = (\omega^2 + \alpha^2)\Phi_{\Delta, \omega}(T, X)$$

- $(0 + 1)$ -dimensional conformal group $SO(2, 1) \cong SL(2, \mathbb{R})/\mathbb{Z}_2$ contains three distinct one-parameter subgroups:
 - ◆ compact rotation group $SO(2)$
 - ◆ noncompact Euclidean group $E(1)$
 - ◆ noncompact Lorentz group $SO(1, 1)$

- Correspondingly, there exist three distinct classes of static AdS_2 coordinate patches in which time-translation generators generate the one-parameter subgroups $SO(2)$, $E(1)$ and $SO(1, 1)$.

- In Lorentzian signature, these coordinate patches are given by the so-called global, Poincaré and Schwarzschild coordinates, respectively.

coordinate patch	time-translation group		spectrum	
	Lorentzian	Euclidean	Lorentzian	Euclidean
global	$SO(2)$	$SO(1, 1)$	discrete	continous
Poincaré	$E(1)$	$E(1)$	continuous	continous
Schwarzschild	$SO(1, 1)$	$SO(2)$	continuous	discrete (Matsubara frequency)

- The Klein-Gordon equation $(\square_{\text{AdS}_2} - m^2)\Phi = 0$ on charged AdS_2 black hole reduces to the following Schrödinger equations:

1. **Global coordinates:** Rosen-Morse potential

$$\left[-\partial_\sigma^2 + \frac{\Delta(\Delta-1)}{\sin^2 \sigma} + 2\omega\alpha \cot \sigma \right] \Phi_{\Delta,\omega} = (\omega^2 - \alpha^2)\Phi_{\Delta,\omega}$$

2. **Poincaré coordinates:** Coulomb potential

$$\left[-\partial_x^2 + \frac{\Delta(\Delta-1)}{x^2} + \frac{2\omega\alpha}{x} \right] \Phi_{\Delta,\omega} = \omega^2 \Phi_{\Delta,\omega}$$

3. **Schwarzschild coordinates:** Eckart potential

$$\left[-\partial_X^2 + \frac{\Delta(\Delta-1)}{\sinh^2 X} + 2\omega\alpha \coth X \right] \Phi_{\Delta,\omega} = (\omega^2 + \alpha^2)\Phi_{\Delta,\omega}$$

Correlator Recurrence Relations

- The Lie algebra $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$ is spanned by the three generators $\{J_1, J_2, J_3\}$ that satisfy the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = -iJ_2$$

- In the Cartan-Weyl basis $\{J_3, J_{\pm} := -J_1 \pm iJ_2\}$ the commutation relations become

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_3$$

- The quadratic Casimir of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = J_3(J_3 \pm 1) - J_{\mp}J_{\pm}$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of C and J_3 that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad J_3|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

Then the state $J_{\pm}|\Delta, \omega\rangle$ satisfies $J_3 J_{\pm}|\Delta, \omega\rangle = (\omega \pm 1)J_{\pm}|\Delta, \omega\rangle$, which implies the ladder equations

$$J_{\pm}|\Delta, \omega\rangle \propto |\Delta, \omega \pm 1\rangle$$

- Let us next consider the following linear combinations

$$L_1 = J_1, \quad L_{\pm} = J_2 \pm J_3$$

which satisfy the commutation relations

$$[L_1, L_{\pm}] = \pm i L_{\pm}, \quad [L_+, L_-] = 2i L_1$$

- The quadratic Casimir of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = -L_1(L_1 \pm i) - L_{\mp} L_{\pm}$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of C and L_1 that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad L_1|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

Then the state $L_{\pm}|\Delta, \omega\rangle$ satisfies $L_1 L_{\pm}|\Delta, \omega\rangle = (\omega \pm i)L_{\pm}|\Delta, \omega\rangle$, which implies the ladder equations

$$L_{\pm}|\Delta, \omega\rangle \propto |\Delta, \omega \pm i\rangle$$

- In the AdS_2 black hole problem, the generators $\{L_1, L_{\pm}\}$ are given by the following first-order differential operators:

$$L_1 = i\partial_T + \alpha$$

$$L_{\pm} = \pm e^{\pm T} \sinh x [\pm i\partial_X + \coth x (i\partial_T + \alpha) - \alpha]$$

- The quadratic Casimir gives the d'Alembertian on AdS_2 black hole:

$$C := L_1(L_1 \pm i) - L_{\mp}L_{\pm} = \sinh^2 X [(i\partial_T^2 + \alpha) + \partial_X^2 + \alpha^2 - 2\alpha \coth X (i\partial_T + \alpha)]$$

- The eigenvalue equations reduce to the Schrödinger equation

$$i\partial_T \Phi_{\Delta, \omega}(T, X) = (\omega - \alpha) \Phi_{\Delta, \omega}(T, X)$$

$$\left(-\partial_X^2 + \frac{\Delta(\Delta - 1)}{\sinh^2 X} + 2\alpha\omega \coth X \right) \Phi_{\Delta, \omega}(T, X) = (\omega^2 + \alpha^2) \Phi_{\Delta, \omega}(T, X)$$

- The ladder equations are

$$L_{\pm} \Phi_{\Delta, \omega} \propto \Phi_{\Delta, \omega \pm i}$$

- In the asymptotic near-boundary limit $X \rightarrow 0$ the generators behave as

$$L_1^0 := \lim_{X \rightarrow 0} L_1 = i\partial_T + \alpha$$

$$L_{\pm}^0 := \lim_{X \rightarrow 0} L_{\pm} = e^{\pm T} [iX\partial_X \pm (i\partial_T + \alpha)]$$

- The quadratic Casimir is

$$C^0 := L_1^0(L_1^0 \pm i) - L_{\mp}^0 L_{\pm}^0 = X^2 \partial_X^2$$

- The eigenvalue equations are

$$i\partial_T \Phi_{\Delta, \omega}^0(T, X) = (\omega - \alpha) \Phi_{\Delta, \omega}^0(T, X)$$

$$\left(-\partial_X^2 + \frac{\Delta(\Delta - 1)}{X^2} \right) \Phi_{\Delta, \omega}^0(T, X) = 0$$

which are easily solved with the result

$$\Phi_{\Delta, \omega}^0(T, X) = A_{\Delta}(\omega) X^{\Delta} e^{-i(\omega - \alpha)T} + B_{\Delta}(\omega) X^{1 - \Delta} e^{-i(\omega - \alpha)T}$$

where $A_{\Delta}(\omega)$ and $B_{\Delta}(\omega)$ are integration constants that depend on Δ and ω .

- The ladder equations $L_{\pm}^0 \Phi_{\Delta, \omega}^0 \propto \Phi_{\Delta, \omega \pm i}^0$ become

$$\begin{aligned} & (i\Delta \pm \omega)A_{\Delta}(\omega)X^{\Delta}e^{-i(\omega \pm i - a)T} + (i(1 - \Delta) \pm \omega)B_{\Delta}(\omega)X^{1 - \Delta}e^{-i(\omega \pm i - a)T} \\ \propto & A_{\Delta}(\omega \pm i)X^{\Delta}e^{-i(\omega \pm i - a)T} + B_{\Delta}(\omega \pm i)X^{1 - \Delta}e^{-i(\omega \pm i - a)T} \end{aligned}$$

from which we get

$$\begin{aligned} (i\Delta \pm \omega)A_{\Delta}(\omega) & \propto A_{\Delta}(\omega \pm i) \\ (i(1 - \Delta) \pm \omega)B_{\Delta}(\omega) & \propto B_{\Delta}(\omega \pm i) \end{aligned}$$

- Hence the Green function

$$G_{\Delta}(\omega) = (2\Delta - 1) \frac{A_{\Delta}(\omega)}{B_{\Delta}(\omega)}$$

satisfies the recurrence relations

$$G_{\Delta}(\omega) = \frac{-1 + \Delta \pm i\omega}{-\Delta \pm i\omega} G_{\Delta}(\omega \pm i)$$

- The recurrence relations

$$G_{\Delta}(\omega) = \frac{-1 + \Delta \pm i\omega}{-\Delta \pm i\omega} G_{\Delta}(\omega \pm i)$$

are easily solved by iteration:

$$G_{\Delta}^{A/R}(\omega) = \frac{\Gamma(\Delta \pm i\omega)}{\Gamma(1 - \Delta \pm i\omega)} g^{A/R}(\Delta)$$

where $g^{A/R}(\Delta)$ are ω -independent normalization factors.

- Shifting the frequency $\omega \rightarrow \omega + \alpha$ and restoring R via $\omega \rightarrow \omega R$, we get the advanced/retarded Green functions

$$G_{\Delta}^{A/R}(\omega) = \frac{\Gamma(\Delta \pm \frac{i\omega}{2\pi T} \pm i\alpha)}{\Gamma(1 - \Delta \pm \frac{i\omega}{2\pi T} \pm i\alpha)} g^{A/R}(\Delta)$$

where T is the Hawking temperature given by

$$T = \frac{1}{2\pi R}$$

Summary and Perspective

Summary

- $SO(2, 1)$ symmetry of AdS_2 black hole induces the recurrence relations for two-point functions:

$$G_{\Delta}(\omega) = \frac{-1 + \Delta \pm i\omega}{-\Delta \pm i\omega} G_{\Delta}(\omega \pm i)$$

- The recurrence relations are exactly solvable and completely determine the frequency dependence of advanced/retarded two-point functions.

Perspective

- Generalizations to AdS_{d+1}/CFT_d for $d > 2$.
(The case $d = 2$ has been done in my previous work [arXiv:1312.7348](https://arxiv.org/abs/1312.7348).)

Thank you for your attention!
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