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A Theory of Clothed Unstable Particles

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The clothed states of unstable particles are investigated on the basis of quantum field theory, and thereby a new mathematical notion, "complex distribution", is introduced. Then the exact eigenstate of total Hamiltonian with the complex eigenvalue, whose real part represents the mass of the unstable particle and whose imaginary part is interpreted as the half reciprocal of its lifetime, can be constructed by means of the complex distribution. But this state is not observable. The physical state of the unstable particle therefore is defined as an approximate state of the exact eigenstate, which exhibits physically reasonable behaviours.

§ 1. Introduction

Recently many authors investigated the theory of unstable particles.[1)–5)] Especially, Araki *et al.*[1)] proposed two methods for naturally defining the physical mass and the lifetime of an unstable particle. The corresponding Z-factors are also defined, but they can take values larger than unity contrary to the stable case. Naito[5)] investigated the production and decay of an unstable particle in the stationary treatment, and defined its physical state, which was formally identical with the approximate eigenstate of total Hamiltonian previously proposed by Glaser and Källén.[3)] He also defined a Z-factor by the normalization constant of this state, which was interpreted as the dissociation probability (of course $0<Z<1$). This Z-factor, however, has a curious property, namely it tends to one half instead of unity at the weak coupling limit, provided that all other interactions are switched off. This fact probably gives rise to theoretical difficulties.* We therefore investigate further the clothed state of an unstable particle.

Now, as a preliminary, we briefly review the second definitions of the mass and life presented by Araki *et al.*, which seem to have the most essential meaning. For simplicity, we employ Lee's model[6)] for the time being. The total Hamiltonian is given by

$$H=m_0\psi_V{}^*\psi_V+m_N\psi_N{}^*\psi_N+\int\omega_k\alpha_k{}^*\alpha_k dk$$
$$+g[\psi_V{}^*\psi_N\int\{G(\omega_k)/\sqrt{2\omega_k}\}\alpha_k dk+\psi_N{}^*\psi_V\int\{G(\omega_k)/\sqrt{2\omega_k}\}\alpha_k{}^*dk]. \quad (1\cdot1)$$

* From this standpoint, the usual perturbational approach to weak interactions would become inadequate. Further, for example, the charge independence would be violated, since proton is stable while neutron is unstable.

Decay amplitude of an unstable particle V:

$$\langle n- | V \rangle = \langle n | U(\infty,0) | V \rangle$$

where

$|n-\rangle$: incoming-wave eigenstate of the *total* Hamiltonian H

$|V\rangle$: ***bare*** state of V

$U(\infty,0)$: wave matrix

S-matrix element of the transition $n \leftarrow m$:

$$\langle n- | m+ \rangle = \langle n | U(\infty,0) \cdot U(0,-\infty) | m \rangle$$

where

$|m+\rangle$: outgoing-wave eigenstate of the *total* Hamiltonian H

$|m\rangle$: ***bare*** state of m

The treatment is quite unsymmetrical !!

If one naively substitutes $|V\rangle$ into $|m\rangle$, then the corresponding S-matrix element *exactly vanishes !!*

Here the notations are as follows.

m_0 : bare mass of a V-particle.

m_N: mass of an N-particle.

μ, $\boldsymbol{k}$ and $\omega_k=\sqrt{\boldsymbol{k}^2+\mu^2}$: mass, momentum and energy of a θ-particle, respectively.

g : unrenormalized coupling constant.

$G(\omega_k)$: cut-off function, which is assumed to be regular.

$\psi_V{}^*$, ψ_V ; $\psi_N{}^*$, ψ_N ; $\alpha_k{}^*$, α_k : field operators of V-particle, N-particle and θ-particle, respectively.

The modified propagator of a V-particle, $S_V{}'(E)$, is given by the following formula:

$$[S_V{}'(E)]^{-1}=E-m_0+i\varepsilon+g^2\int\frac{G^2(\omega_k)}{2\omega_k}\cdot\frac{d\boldsymbol{k}}{\omega_k+m_N-E-i\varepsilon},\quad(\varepsilon\to+0).\qquad(1\cdot2)$$

If the equation which is obtained by putting the right-hand side of $(1\cdot2)$ equal to zero has a real root, say, $E=m$, then m represents the mass of the stable V-particle because of $m<m_N+\mu$ which is evident from the integral in $(1\cdot2)$. So, in order to obtain the mass of the unstable V-particle, we must analytically continue $[S_V{}'(E)]^{-1}$ to the complex plane of E. But $[S_V{}'(E)]^{-1}$ has no zero point on the Riemann plane with cut along the real axis from the branching point, $m_N+\mu$, to $+\infty$. Hence we consider the analytic continuation to the lower half-plane from the side of Re $E>m_N+\mu$. Then for E on the lower half-plane the path of ω_k-integration must be deformed as in Fig. 1 on the ω_k-plane. The integral therefore is written as the sum of the integral along the real axis and of the residue of a pole, $\omega_k=E-m_N$, that is to say,

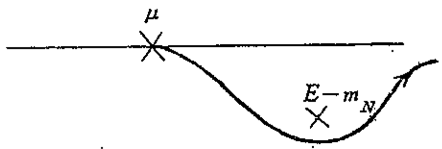

Fig. 1

$$[S_V{}'(E)]^{-1}=E-m_0+2\pi g^2\int_\mu^{+\infty}\frac{G^2(\omega)\sqrt{\omega^2-\mu^2}}{\omega+m_N-E}d\omega$$
$$+(2\pi)^2 ig^2G^2(E-m_N)\sqrt{(E-m_N)^2-\mu^2},\quad(\text{for Im}\,E<0).\qquad(1\cdot3)$$

The equation which is obtained by putting the right-hand side of $(1\cdot3)$ equal to zero generally has complex roots. For simplicity, we assume that the equation has only one root, say, $E=m_V-i\gamma/2$, *i. e.*

$$[S_V{}'(m_V-i\gamma/2)]^{-1}=0.\qquad(1\cdot4)$$

m_V and γ^{-1} are interpreted as the physical mass and the life of the unstable V-particle, respectively.

Now, in the case of the stable particle, its mass is defined not only as the pole of modified propagator like $(1\cdot4)$ but also as the one-particle-state eigenvalue of total Hamiltonian. For the unstable particle, however, such an eigenstate does not exist, since the scattering states alone form a complete set.[1),3)] This corresponds to that $(1\cdot2)$ has no zero point. But as we have obtained a zero point by the analytic continuation like

(1·3), there should exist the corresponding eigenstate. To get this state needs, so to speak, the analytic continuation of state-vector, which is not a known notion. So we introduce a new notion, "*complex distribution*", in the next section. In § 3, we explicitly construct the eigenstate of total Hamiltonian with the eigenvalue $m_V - i\gamma/2$ by using the complex distribution. But this state is not contained in the conventional Hilbert space and is not observable. In § 4, we define the physical state of the unstable V-particle by an approximate one to the above eigenstate, and show that its properties are physically reasonable. In § 5, generalizations are presented. Finally, we discuss the fundamental postulate of quantum field theory in the light of the complex distribution.

§ 2. Definition of complex distribution

Consider a meromorphic function, $F(\omega)$, and two fixed points, a and b, in a domain D. For any arbitrary function, $\varphi(\omega)$, regular in D, we consider an integral,

$$F[\varphi] \equiv \int_a^b \varphi(\omega) F(\omega) d\omega,$$

as a linear functional of $\varphi(\omega)$. This functional is *not* well defined by $F(\omega)$ *alone*, but we must further indicate which side of each pole of $F(\omega)$ the path of integration passes through. We call the meromorphic function with such indications "complex distribution" as a generalization of Schwartz's distribution.[7]

In the following we choose μ and $+\infty$ as the end points of the path. The indications for choosing the path are denoted by the following notations.

$$\left.\begin{array}{ll} 1/(\omega^{(+)}-c): & \text{The path passes through } above \text{ the pole } \omega=c. \\ 1/(\omega^{(-)}-c): & \text{The path passes through } below \text{ the pole } \omega=c. \end{array}\right\} \tag{2·1}$$

Then we evidently have*

$$\left.\begin{array}{l} \{1/(\omega^{(+)}-c)\}^* = 1/(\omega^{(-)}-c^*), \\ \{1/(\omega^{(-)}-c)\}^* = 1/(\omega^{(+)}-c^*) \end{array}\right\} \tag{2·2}$$

for complex conjugation. The difference between the both of (2·1) is nothing but the contribution from the pole. We therefore define complex δ-function as follows:

$$2\pi i \delta(\omega - c) \equiv 1/(\omega^{(-)}-c) - 1/(\omega^{(+)}-c). \tag{2·3}$$

From (2·3) we immediately get

$$\int_\mu^{+\infty} \varphi(\omega)\, \delta(\omega - c)\, d\omega = \varphi(c), \tag{2·4}$$

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* This is because

$$\left[\int d\omega\, \varphi(\omega)/(\omega^{(+)}-c)\right]^* = \int d\omega\, \varphi^*(\omega)/(\omega^{(-)}-c^*).$$

It must be noticed that ω is not a definite complex number but an analytic variable, and Re ω, Im ω and ω^* are meaningless.

and

$$[\delta(\omega-c)]^*=\delta(\omega-c^*) \tag{2·5}$$

because of (2·2). Hence we have

$$\int_{\mu}^{+\infty}[\delta(\omega-c)]^*\delta(\omega-c)\,d\omega=0, \quad (\text{for Im } c\neq 0). \tag{2·6}$$

The definitions of the derivatives of the complex δ-function are straightforward. *Complex distributions can generally be represented by usual functions and complex δ-functions by using* (2·3).

In the above, complex distributions are defined only for the integrations from μ to $+\infty$. Though no other integrations are needed in the next section, it is necessary for the general case that complex distribiuton should be defined for the more general integrations. A convenient generalization to the integrations between two real points, a and b, is as follows: the path runs according to its indication only for the each pole whose real part is between a and b. Then the simple additivity of integrations is satisfied. In particular, consider the case in which c in (2·1) is real. The complex distribution then reduces to the well-known distribution, $1/(\omega-c\pm i\varepsilon)$, $(\varepsilon\to+0)$. The former thus is a straightforward generalization of the latter to the case in which the pole is apart from the real axis by a finite distance.

Finally, in the case of two analytic variables a complex distribution, $1/(\omega_1^{(-)}-\omega_2^{(+)})$, is defined as follows:

$$\int\frac{\varphi(\omega_1,\ \omega_2)}{\omega_1^{(-)}-\omega_2^{(+)}}d\omega_1 d\omega_2\equiv\int\frac{\varphi(\omega_1,\ \omega_2)}{\omega_1-\omega_2-i\varepsilon}d\omega_1 d\omega_2, \tag{2·7}$$

for any analytic *function*, $\varphi(\omega_1,\ \omega_2)$, and

$$\left.\begin{aligned}\delta(\omega_1-c)/(\omega_1^{(-)}-\omega_2^{(+)})&\equiv\delta(\omega_1-c)/(c-\omega_2^{(+)}),\\ \delta(\omega_2-c)/(\omega_1^{(-)}-\omega_2^{(+)})&\equiv\delta(\omega_2-c)/(\omega_1^{(-)}-c)\end{aligned}\right\} \tag{2·8}$$

for complex δ-functions and likewise for their derivatives. The following identities then can be easily verified.

$$\frac{1}{(\omega_1^{(-)}-c)(\omega_1^{(-)}-\omega_2^{(+)})}=-\frac{1}{\omega_2^{(+)}-c}\left(\frac{1}{\omega_1^{(-)}-c}-\frac{1}{\omega_1^{(-)}-\omega_2^{(+)}}\right), \tag{2·9}$$

$$\frac{1}{(\omega_1^{(-)}-c)(\omega_1^{(+)}-\omega_2^{(-)})}=-\frac{1}{\omega_2^{(-)}-c}\left(\frac{1}{\omega_1^{(-)}-c}-\frac{1}{\omega_1^{(+)}-\omega_2^{(-)}}\right). \tag{2·10}$$

§ 3. Exact state of the unstable V-particle

The physical state of the *stable* V-particle is defined as the exact eigenstate of H with the eigenvalue m, namely

$$|V\rangle-g\int\frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k+m_N-m}\alpha_k{}^*d\boldsymbol{k}\,|N\rangle, \tag{3·1}$$

where $|V\rangle$ and $|N\rangle$ respectively stand for the bare-V state and the N state.

In the case of the *unstable* V-particle, we obtain the state proposed by Naito[5] if we formally put $m=m_V-i\gamma/2$ in (3·1). But this is, of course, *not* the eigenstate of H. This is because the Riemann plane in (1·3) differs from that in (3·1). To get the former it is necessary to curve the path of the ω_k-integration like in Fig. 1. But it is meaningless to curve that in (3·1), since this integration is not mathematical but purely formal one because of the presence of $\alpha_k{}^*$. So we make use of the complex distribution defined in last section in order to indicate the path potentially, that is to say, we introduce a state

$$|\boldsymbol{V}\rangle = |V\rangle - g\int \frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(-)}+m_N-m_V+i\gamma/2}\alpha_k{}^* d\boldsymbol{k}\,|N\rangle. \tag{3·2}$$

Indeed, this state exactly satisfy the equation,*

$$H|\boldsymbol{V}\rangle = (m_V-i\gamma/2)|\boldsymbol{V}\rangle. \tag{3·3}$$

At first sight (3·3) seems very curious, for H is an Hermitian operator.** Actually, from (3·3) we obtain

$$i\gamma\langle\boldsymbol{V}|\boldsymbol{V}\rangle = \langle\boldsymbol{V}|(H^*-H)|\boldsymbol{V}\rangle = 0. \tag{3·4}$$

Hence if $|\boldsymbol{V}\rangle$ had a positive norm, γ would have to vanish. In the present case we must have

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle = 0 \tag{3·5}$$

because of $\gamma\neq 0$. (3·5) is verified also by the following direct calculation. From (2·2) we have

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle = 1+g^2\int\frac{G^2(\omega_k)}{2\omega_k}\cdot\frac{d\boldsymbol{k}}{(\omega_k^{(+)}+m_N-m_V-i\gamma/2)(\omega_k^{(-)}+m_N-m_V+i\gamma/2)}. \tag{3·6}$$

The path of the integration is shown in Fig. 2, that is, (3·6) is rewritten as

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle = 1+g^2\int_{\text{real axis}}\frac{G^2(\omega_k)}{2\omega_k}\cdot\frac{d\boldsymbol{k}}{(\omega_k+m_N-m_V)^2+(\gamma/2)^2}$$
$$-2(2\pi)^2 g^2\,\mathrm{Re}\left[G^2(m_V-m_N-i\gamma/2)\sqrt{(m_V-m_N-i\gamma/2)^2-\mu^2}\,\right]/\gamma. \tag{3·7}$$

Making use of (1·3) we have

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* Though the commutation relation naturally is $[\alpha_k, \alpha_{k'}{}^*]=\delta(\boldsymbol{k}-\boldsymbol{k}')$, this is no longer the usual δ-function, but the complex distribution defined by

$$\delta(\boldsymbol{k}-\boldsymbol{k}') = (k\omega_k)^{-1}\delta(\omega_k-\omega_k{}')\,\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi', \text{ (mod. } 2\pi)),$$

where k, θ, φ and k', θ', φ' are the polar coordinates of $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. But this expression is unnecessary in the actual calculations.

** Strictly speaking, the Hermiticity of an operator is meaningful only when its operand is designated. We have extended not the basic vectors ($|V\rangle$, $\alpha_k{}^*|N\rangle$) but the distributions as their coefficients. Since we have defined the complex distribution so as to be consistent with the usual complex conjugation, the Hermiticity of H (in the generalized sense) is not injured at all.

$$\langle V|V\rangle = -(\gamma/2)^{-1}\,\mathrm{Im}[S_V'(m_V - i\gamma/2)]^{-1}. \qquad (3\cdot 8)$$

We thus obtain (3·5) because of (1·4).

Thus there has appeared a zero-norm state in our Hilbert space. Our Hilbert space is a little wider than the conventional one, since the former contains negative-norm states. But it is not represented by the well-known indefinite-metric device.[8] Incidentally, the positive-norm restriction is very additional in the mathematical definition of the abstract Hilbert space. So the general Hilbert space with indefinite norm is not necessarily represented by indefinite metric. Indeed, our Hilbert space is such an example, and we can construct a consistent theory.

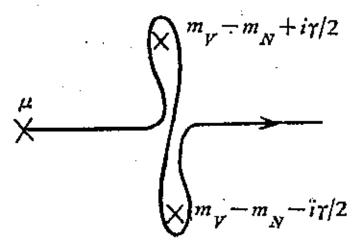

Fig. 2

In the unstable case we know that N-θ scattering states form a complete set. But as $|V\rangle$ is a new eigenstate of H, its expansions by complete sets are of interest.

(i) Incoming states

N-θ incoming-wave eigenstates are given by

$$|N\theta(\boldsymbol{p})^-\rangle \equiv g\frac{G(\omega_p)}{\sqrt{2\omega_p}}S_V'^*(m_N+\omega_p^{(-)})|V\rangle$$
$$+\int\left[\delta(\boldsymbol{k}-\boldsymbol{p}) - g\frac{G(\omega_p)}{\sqrt{2\omega_p}}S_V'^*(m_N+\omega_p^{(-)})\frac{gG(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(+)}-\omega_p^{(-)}}\right]\alpha_k^*\,d\boldsymbol{k}\,|N\rangle. \qquad (3\cdot 9)$$

Here the complex-distribution notation is used instead of the usual $-i\varepsilon$ convention in the denominator of the last term.* The modified propagator, $S_V'(E)$, likewise is written as

$$[S_V'(E)]^{-1} = E - m_0 + g^2\int\frac{G^2(\omega_k)/2\omega_k}{\omega_k^{(-)}+m_N-E}d\boldsymbol{k} \qquad (3\cdot 10)$$

in terms of complex distribution. This is equal to (1·2) for $\mathrm{Im}\,E \geqq 0$ and to (1·3) for $\mathrm{Im}\,E<0$. A straightforward calculation of $\langle N\theta(\boldsymbol{p})^-|V\rangle$ yields

$$\langle N\theta(\boldsymbol{p})^-|V\rangle = -2\pi i g\{G(\omega_p)/\sqrt{2\omega_p}\}\,\delta(\omega_p+m_N-m_V+i\gamma/2), \qquad (3\cdot 11)$$

where we have made use of (2·9), (3·10), (1·4) and (2·3). (3·11) naturally is consistent with

$$(\omega_p+m_N-m_V+i\gamma/2)\langle N\theta(\boldsymbol{p})^-|V\rangle = 0, \qquad (3\cdot 12)$$

which follows from

$$H|N\theta(\boldsymbol{p})^-\rangle = (m_N+\omega_p)|N\theta(\boldsymbol{p})^-\rangle \qquad (3\cdot 13)$$

and (3·3). From (3·11) $|V\rangle$ is expanded into

* This replacement naturally does not change the usual properties of this state, *e.g.* its normalization.

$$|V\rangle = -2\pi i g\int \frac{G(\omega_p)}{\sqrt{2\omega_p}}\delta(\omega_p+m_N-m_V+i\gamma/2)\,|N\theta(\boldsymbol{p})^-\rangle\,d\boldsymbol{p} \tag{3·14}$$

in terms of the complete set of $|N\theta(\boldsymbol{p})^-\rangle$. (3·14) is verified also by substituting (3·9) in its right-hand side, where a formula,

$$S_V'^*(m_V-i\gamma/2) = -[2\pi i g^2\int \frac{G^2(\omega_k)}{2\omega_k}\delta(\omega_k+m_N-m_V+i\gamma/2)\,d\boldsymbol{k}]^{-1}, \tag{3·15}$$

which follows from

$$[S_V'^*(E)]^{-1} = [S_V'(E)]^{-1} - 2\pi i g^2\int \frac{G^2(\omega_k)}{2\omega_k}\delta(\omega_k+m_N-E)\,d\boldsymbol{k} \tag{3·16}$$

and (1·4), should be taken into account.

(ii) Out-going states

The out-going states similarly are

$$|N\theta(\boldsymbol{p})^+\rangle \equiv g\frac{G(\omega_p)}{\sqrt{2\omega_p}}S_V'(m_N+\omega_p^{(+)})\,|V\rangle$$
$$+\int\left[\delta(\boldsymbol{k}-\boldsymbol{p}) - g\frac{G(\omega_p)}{\sqrt{2\omega_p}}S_V'(m_N+\omega_p^{(+)})\frac{gG(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(-)}-\omega_p^{(+)}}\right]\alpha_k^*\,d\boldsymbol{k}|N\rangle. \tag{3·17}$$

Since $|N\theta(\boldsymbol{p})^+\rangle$'s are also the eigenstates of H, it should be

$$\langle N\theta(\boldsymbol{p})^+|V\rangle \propto \delta(\omega_p+m_N-m_V+i\gamma/2)$$

like the above. The coefficient, however, turns out to be equal to zero. This is because the δ-singularity coincides with the singular point of (3·17), and so the expansion corresponding to (3·14) becomes an indefinite form, $0\times\infty$. This is not a defect characteristic of complex distribution, but a general character of distribution. Indeed, similar singular characters often appear in the conventional scattering problems. To avoid this difficulty, the states containing a complex parameter E,

$$|V_E\rangle \equiv |V\rangle - g\int \frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(-)}+m_N-E}\alpha_k^*\,d\boldsymbol{k}\,|N\rangle, \tag{3·18}$$

whose norm is $-(2/\gamma)\,\mathrm{Im}\,[S_V'(E)]^{-1}$, are introduced. Then making use of (2·10) we have

$$\langle N\theta(\boldsymbol{p})^+|V_E\rangle = -g\frac{G(\omega_p)/\sqrt{2\omega_p}}{\omega_p^{(-)}+m_N-E}S_V'^*(m_N+\omega_p^{(-)})[S_V'(E)]^{-1}, \tag{3·19}$$

for which we obtain meaningful expansions.

Now, $|\langle N\theta(\boldsymbol{p})^-|V\rangle|^2$ should be the decay spectrum of $|V\rangle$ apart from the phase-volume factor, but it turns out to vanish from (3·11) and (2·6). This is obvious from the time-dependent Schrödinger equation,

$$i(\partial/\partial t)\Psi(t) = H\Psi(t), \tag{3·20}$$

with $\Psi(0)=|V\rangle$ as the initial condition. From (3·3) we have

$$\Psi(t)=|V\rangle e^{-i(m_V-i\gamma/2)t} \tag{3·21}$$

which monotonously diminishes. But since the norm of $\Psi(t)$ is always equal to zero, no inconsistency appears. $|V\rangle$ gives rise to no difficulty about the conservation of probability contrary to the so-called ghost state.[9]

$|V\rangle$ thus is not observable, and therefore should not be called "physical state". So we call it "exact state of the unstable V-particle".

§ 4. Physical state of the unstable V-particle

The exact state introduced in last section is never observable, as far as one adopts the present theory of observation. Namely, $|V\rangle$ is a state beyond the "observation space" (*i. e.* conventional Hibert space). So the physical decaying state of the unstable V-particle should be regarded as an image of $|V\rangle$ projected into the observation space. Under this idea, a physical state is proposed as an approximate state to $|V\rangle$ with a positive norm in this section.

The very origin of the zero-norm is nothing but the complex δ-function in (3·2). This circumstance is caused essentially by $\int[\delta(\omega_k+m_N-m_V+i\gamma/2)]^*\cdot\delta(\omega_k+m_N-m_V+i\gamma/2)d\omega_k=0$, which would never hold if it were a function. Hence to obtain the positive-norm state we have only to replace this complex δ-function by a usual function. $2\pi i\cdot\delta(\omega_k+m_N-m_V+i\gamma/2)$ is just the contribution from the pole of $1/(\omega_k+m_N-m_V+i\gamma/2)$, but the behaviour of the both are completely different away from the pole. Hence we consider a replacement,

$$2\pi i\delta(\omega_k+m_N-m_V+i\gamma/2)\to -f(\omega_k)/(\omega_k+m_N-m_V+i\gamma/2), \tag{4·1}$$

where $f(\omega)$ is a single-valued, analytic function satisfying the following conditions.

1) $f(\omega)\approx 1$ for $|\omega+m_N-m_V|\ll\kappa$,
where $m_V-m_N-\mu\gg\kappa\gg\gamma$.
2) $f(\omega)\approx 0$ effectively, for $|\omega+m_N-m_V|\gg\kappa$,
where ω is real.
3) regular near the real axis.
4) vanishing far away in the lower half-plane.

Here we have introduced a new parameter, κ, so that the value of $f(\omega)$ at the very pole and that near $\omega=m_V-m_N$ on the real axis be not appreciably different. The reason for the minus sign in (4·1) is that the integral along the real axis is effectively equivalent to one round of the pole with the negative direction (cf. conditions 3) and 4)).

A typical example of $f(\omega)$ is

$$f(\omega)=\{\lambda^2/(\omega^2+\lambda^2)\}e^{-i(\omega+m_N-m_V)/\kappa}, \tag{4·2}$$

where $m_V\ll\lambda\ll\Lambda$, and Λ is the cut-off parameter in $G(\omega)$. When $G(\omega)$ is present, the factor $\lambda^2/(\omega^2+\lambda^2)$ is unnecessary. If the limits of $\gamma\to 0$, $\kappa\to 0$ and $\lambda\to\infty$ are taken generally be controlled by experiments, and so the calculations become futilely complicated. We therefore propose to use the exact state instead of the physical one in the calculations including unstable particles. According to this method, we can treat transition matrices of the decay or scattering of unstable particles in the same way as in the usual S-matrix theory of stable particles. This method thus permits to treat unstable particles by the S-matrix-theoretical formalism.

§ 2. Physical state

In this section we employ Lee's model[4] for the sake of simplicity. Notations are same as in I throughout this paper, unless otherwise indicated.

According to the scattering formalism of Gell-Mann and Goldberger, the outgoing-wave eigenstate is obtained by the following adiabatic process:

$$|N\theta(\boldsymbol{p})^+\rangle = \lim_{\varepsilon\to+0}\int_{-\infty}^{0} dT\ \varepsilon e^{\varepsilon T} e^{i(H-m_N-\omega_p)T} \alpha_p^* |N\rangle\,. \qquad (2\cdot1)$$

This represents a superposition of the incident waves emitted in various times at the presence of interaction. Quite analogously to this, let us consider the following state:

$$\Psi^\kappa \equiv \int_{-\infty}^{0} dT\ \eta_\kappa(T) e^{i(H-E_0)T} |V\rangle\,, \qquad (2\cdot2)$$

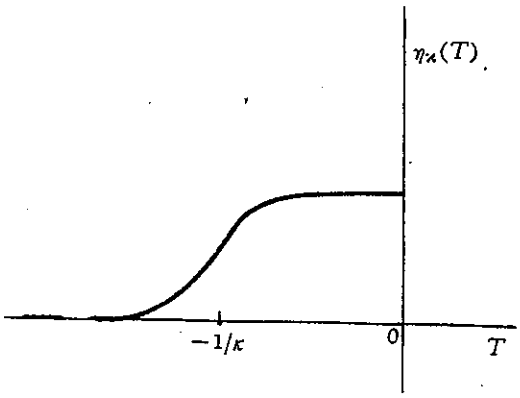

Fig. 1.

which represents the superposition of the V-particle amplitudes produced with intensity $\eta_\kappa(T)$. Actually, the physical V-particle is produced in a finite time interval, and so the state Ψ^κ defined above may be regarded as the physical unstable state. Here we assume that the function $\eta_\kappa(T)$ roughly takes such a form as Fig. 1. As κ must be finite, Ψ^κ generally depends on the form of the function $\eta_\kappa(T)$. The usual choice

$$\eta_\kappa(T) = \kappa e^{\kappa T} \qquad (2\cdot3)$$

is not preferable, because it has the following defects.

i) It has a finite tangent at the final time $T=0$.*

ii) It does not damp sufficiently rapidly for $T\to-\infty$.

The meaning of these respects will be clarified later. E_0, which represents the rest energy of the V-particle, might be m_0 for the bare state, but we should use here the renormalized mass as E_0, as is well known. Namely, we put

$$E_0 = m_V - i\gamma/2 \qquad (2\cdot4)$$

* This physically means that the production does not become stationary at the final time.

which is the pole of the modified propagator $S_V'(E)$. Hence $\eta_\varkappa(T)$ must damp faster than at least $e^{\gamma T/2}$ in order to make the integral (2·2) converge. In order to calculate (2·2), we insert into it the complete set[6,8]

$$|N\theta(\mathbf{p})^+\rangle = g(\omega_p) S_V'(m_N+\omega_p)|V\rangle + \int\left[\delta(\mathbf{k}-\mathbf{p}) - g(\omega_p) S_V'(m_N+\omega_p)\frac{g(\omega_k)}{\omega_k-\omega_p-i\varepsilon}\right]\alpha_k^* d\mathbf{k}|N\rangle \tag{2·5}$$

with

$$g(\omega_p) \equiv g\cdot G(\omega_p)/\sqrt{2\omega_p}\,.$$

We then obtain

$$\Psi^\varkappa = \int_{-\infty}^0 \eta_\varkappa(T)dT\Big[\int d\mathbf{p}\,\rho(\omega_p)\,e^{i(m_N+\omega_p-E_0)T}|V\rangle + \int d\mathbf{k}\; g(\omega_k)S_V'^*(m_N+\omega_k)\,e^{i(m_N+\omega_k-E_0)T}\alpha_k^* d\mathbf{k}|N\rangle - \int d\mathbf{p}\cdot\rho(\omega_p)\,e^{i(m_N+\omega_p-E_0)T}\int\frac{g(\omega_k)}{\omega_k-\omega_p-i\varepsilon}\alpha_k^* d\mathbf{k}|N\rangle\Big], \tag{2·6}$$

where

$$\rho(\omega_p) \equiv g^2(\omega_p)S_V'^*(m_N+\omega_p)S_V'(m_N+\omega_p). \tag{2·7}$$

By using Lehmann's spectral representation[5]

$$S_V'^*(m_N+\omega_k) = \int d\mathbf{p}\,\frac{\rho(\omega_p)}{\omega_k-\omega_p-i\varepsilon}, \tag{2·8}$$

(2.6) can be rewritten as

$$\Psi^\varkappa = c|V\rangle + \int d\mathbf{p}\,\rho(\omega_p)\int F(\omega_k,\,\omega_p)\;\; g(\omega_k)\alpha_k^* d\mathbf{k}|N\rangle\,, \tag{2·9}$$

where

$$c \equiv \int d\mathbf{p}\,\rho(\omega_p)\int_{-\infty}^0 dT\,\eta_\varkappa(T)\,e^{i(m_N+\omega_p-E_0)T} \tag{2·10}$$

and

$$F(\omega_k,\,\omega_p) \equiv \int_{-\infty}^0 \eta_\varkappa(T)dT\frac{e^{i(m_N+\omega_k-E_0)T}-e^{i(m_N+\omega_p-E_0)T}}{\omega_k-\omega_p-i\varepsilon}. \tag{2·11}$$

$F(\omega_k,\,\omega_p)$ is regular at $\omega_k=\omega_p$ and so $-i\varepsilon$ in the denominator is unnecessary. $F(\omega_k,\omega_p)$ is rewritten as follows.

$$F(\omega_k,\,\omega_p) = -i\int_{-\infty}^0 \eta_\varkappa(T)\,e^{i(m_N+\omega_p-E_0)T}dT\int_T^0 e^{i(\omega_k-\omega_p)\tau}d\tau$$

$$= -i\int_{-\infty}^0 e^{i(m_N+\omega_k-E_0)\tau}d\tau\int_{-\infty}^\tau \eta_\varkappa(T)\,e^{i(m_N+\omega_p-E_0)(T-\tau)}dT$$

$$=-i\int_{-\infty}^{0}dT\, e^{i(m_N+\omega_p-E_0)T}\int_{-\infty}^{0}d\tau\, e^{i(m_N+\omega_k-E_0)\tau}\eta_\kappa(T+\tau). \qquad (2\cdot12)$$

Substituting (2.12) in (2.9) and integrating by parts relating to τ, we obtain

$$\Psi^\kappa=c|V\rangle-\int d\boldsymbol{p}\,\rho(\omega_p)\int_{-\infty}^{0}dT\, e^{i(m_N+\omega_p-E_0)T}\int d\boldsymbol{k}\frac{g(\omega_k)}{m_N+\omega_k-E_0}$$

$$\cdot\left[\eta_\kappa(T)-\int_{-\infty}^{0}d\tau\, e^{i(m_N+\omega_k-E_0)\tau}\eta_\kappa'(T+\tau)\right]\alpha_k^*|N\rangle$$

$$=c\left[|V\rangle-\int\frac{g(\omega_k)}{m_N+\omega_k-E_0}\{1-f(\omega_k)\}\,\alpha_k^*d\boldsymbol{k}|N\rangle\right], \qquad (2\cdot13)$$

where

$$f(\omega_k)\equiv\int_0^{+\infty}e^{-i(m_N+\omega_k-m_V)t}h(t)\,dt \qquad (2\cdot14)$$

with

$$h(t)\equiv c^{-1}\,e^{\gamma t/2}\int d\boldsymbol{p}\,\rho(\omega_p)\int_{-\infty}^{0}dT\,\eta_\kappa'(T-t)\,e^{i(m_N+\omega_p-E_0)T}. \qquad (2\cdot15)$$

Here the function $h(t)$ has the following properties:

i) $$\int_0^{+\infty}h(t)\,dt=c^{-1}\int d\boldsymbol{p}\;\rho(\omega_p)\int_{-\infty}^{0}dT\left[\eta_\kappa(T)+\int_0^{+\infty}(\gamma/2)\,e^{\gamma t/2}\eta_\kappa(T-t)\,dt\right]e^{i(m_N+\omega_p-E_0)T}$$

$$=1+O(\gamma/\kappa)\,, \qquad (2\cdot16)$$

ii) $$h(0)=c^{-1}\int d\boldsymbol{p}\,\rho(\omega_p)\int_{-\infty}^{0}dT\,\eta_\kappa'(T)\,e^{i(m_N+\omega_p-E_0)T}\approx0. \qquad (2\cdot17)$$

Because $\eta_\kappa'(T)$ is a sufficiently smooth function and has the values appreciably different from zero only in the neighbourhood of $T=-1/\kappa$, and so the integral approximately vanishes due to the rapidly oscillating factor. For $h'(0)$, $h''(0)$, etc., the same is true, provided that $\eta_\kappa(T)$ has the higher derivatives.

Now the final expression of Ψ^κ, (2·13), has completely the same form with the physical state $|V^f\rangle$ given in I, apart from the normalization factor. And, further, we can prove that $f(\omega_k)$ actually satisfies the four conditions postulated in I.

i) $f(\omega_k)\approx1$ for $|\omega_k+m_N-m_V|\ll\kappa$.
Because we then get $f(\omega_k)\approx\int_0^{+\infty}h(t)\,dt\approx1$ from (2·16) for $\gamma\ll\kappa$.

ii) $f(\omega_k)\approx0$ for $|\omega_k+m_N-m_V|\gg\kappa$ where ω_k is real.
Because, $f(\omega_k)$ then vanishes approximately owing to (2·17) for the contribution from $t\approx0$ and to the rapidly oscillating factor for that from the other part.

iii) $f(\omega_k)$ is regular near the real axis.
The analyticity in the lower half-plane is obvious from (2.14). Further, the

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A Note on the Physical State of Unstable Particles

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Recently, we have investigated the clothed states of unstable particles.[1),2)] In the second paper,[2)] we have shown that the physical state of the unstable V-particle is given by

$$|\boldsymbol{V}^{\times}\rangle \equiv N \cdot F_{\times}(H)|V\rangle \qquad (1)$$

with

$$F_{\times}(H) \equiv \int_{-\infty}^{0} dT\,\eta_{\times}(T)\, e^{i(H-E_0)T}, \qquad (2)$$

where the notations are as follows:

$|V\rangle$: bare state,

H: total Hamiltonian,

$E_0 = m_V - i\gamma/2$,

$\eta_{\times}(T)$: production-rate function,[2)]

N: normalization constant.

Though we have so far used mainly the expression of the physical state, it is more convenient to use the expression (1) in the calculation of various quantities. For example, the decay amplitude is simply presented by

$$\langle \boldsymbol{n}^- | \boldsymbol{V}^{\times}\rangle = N \cdot F_{\times}(E_n)\langle \boldsymbol{n}^- | V\rangle, \qquad (3)$$

where $|\boldsymbol{n}^-\rangle$ is the incoming-wave eigenstate: $H|\boldsymbol{n}^-\rangle = E_n|\boldsymbol{n}^-\rangle$. $\langle \boldsymbol{n}^- | V\rangle$ is just the decay amplitude of the bare V-particle, and the function $F_{\times}(E_n)$ is easily calculated (at least numerically) for a given $\eta_{\times}(T)$. The normalization constant N is given by

$$N^{-2} = \langle V | F_{\times}^{*}(H) F_{\times}(H) | V\rangle = \int dE \rho(E) |F_{\times}(E)|^2, \qquad (4)$$

where

$$\rho(E) \equiv \int dn\, \delta(E - E_n) |\langle V | \boldsymbol{n}^-\rangle|^2 \qquad (5)$$

is Lehmann's spectral function.[3)] Similarly, the time-development of $|\boldsymbol{V}^{\times}\rangle$ is easily calculated, namely

$$\langle \boldsymbol{V}^{\times} | e^{-iHt} | \boldsymbol{V}^{\times}\rangle = N^2 \int dE \rho(E) |F_{\times}(E)|^2 e^{-iEt} \quad \text{for} \quad t \geqq 0. \qquad (6)$$

The main contribution of the integral comes out from the pole $E = E_0$ of $\rho(E)$ in the lower half-plane. Since $F_{\times}(E_0) = \int_{-\infty}^{0} dT \eta_{\times}(T) \equiv 1$, the integral for $E \approx E_0$ is essentially the modified propagator $S_V'(t)$. We therefore obtain[4)]

$$\langle \boldsymbol{V}^{\times} | e^{-iHt} | \boldsymbol{V}^{\times}\rangle \approx N^2 Z_S e^{-iE_0 t}, \qquad (7)$$

where

$$Z_S^{-1} = (\partial/\partial E)[S_V'(E)]^{-1}|_{E=E_0}.$$

Finally, we consider the renormalization. The Z-factor is defined by the probability of the bare V-particle, namely

$$Z \equiv |\langle V | \boldsymbol{V}^{\times}\rangle|^2 = N^2 \left| \int dE \rho(E) F_{\times}(E) \right|^2. \qquad (8)$$

We denote the renormalized quantities by affixing the suffix R. Then we have

$$\langle \boldsymbol{n}^- | V\rangle_R = Z^{-1/2} \langle \boldsymbol{n}^- | V\rangle, \quad \rho_R(E) = Z^{-1}\rho(E), \qquad (9)$$

and (4) and (8) are rewritten as

$$(ZN^2)^{-1}=\int dE\,\rho_R(E)\,|F_\varkappa(E)|^2$$
$$=|\int dE\,\rho_R(E)\,F_\varkappa(E)\,|^2. \tag{10}$$

The decay amplitude can, of course, be expressed by the renormalized quantities alone :

$$\langle \boldsymbol{n}^-|V^\varkappa\rangle = c\cdot F_\varkappa(E_n)\langle \boldsymbol{n}^-|V\rangle_R, \tag{11}$$

where $c^2 \equiv ZN^2$ is given by (10). The production-rate function $\eta_\varkappa(T)$ can be calculated, at least in principle, by (11) from the theoretical value of $\langle \boldsymbol{n}^-|V\rangle_R$ and the experimental decay spectrum (provided that $\eta_\varkappa(T)$ is real).

1) N. Nakanishi, Prog. Theor. Phys. **19** (1958), 607.
2) N. Nakanishi, Prog. Theor. Phys. **20** (1958), to be published.
3) H. Lehmann, Nuovo Cim. **11** (1954), 342.
4) H. Araki, Y. Munakata, M. Kawaguchi and T. Goto, Prog. Theor. Phys. **17** (1957), 419, (Appendix A).