How to parameterize a resonance with finite width (the σ resonance)?

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⋆ The Breit–Wigner description of resonance

In principal the Breit–Wigner description of resonance only works for infinitely small width.

For stable particle,

$$\triangle(p^2) = \frac{1}{p^2 - m^2 + i\epsilon} , \tag{1}$$

Example:

$$A \to B + C, \;\; B \to D + E \; . \tag{2}$$

$$H_{eff} = fABC + gBDE \; , \tag{3}$$

$$\Rightarrow \Gamma_A = \frac{f^2}{16\pi}\frac{m_A^2 - m_B^2}{m_A^3} \; , \;\; \Gamma_B = \frac{g^2}{16\pi}\frac{1}{m_B} \tag{4}$$

Ansatz:

$$\triangle_B(p^2) = \frac{1}{p^2 - \alpha + i\beta} \; . \tag{5}$$

Equivalently, Eq. (2)$\equiv$ $A \to C + D + E$, Now calculating Γ_A (in the calculation $m_A >> \Gamma_A$ is assumed!!!)

$$\Gamma_A = \frac{f^2}{16\pi}\frac{g^2}{16\pi}\frac{m_A^2-\alpha}{m_A^3\beta}. \tag{6}$$

$$\Rightarrow \triangle_{BW}(p^2) = \frac{1}{p^2-m^2+im\Gamma} . \tag{7}$$

Not very good analytical property!

$$\Rightarrow \triangle_{BW}(p^2) = \frac{1}{p^2-m^2+i\rho(s)G} . \tag{8}$$

The σ resonance is a very broad subject!

$$M_\sigma \simeq 500MeV \ , \ \ \Gamma_\sigma \simeq 640MeV \ . \tag{9}$$

Z. G. Xiao and H.Z., Nucl. Phys. A695(2001)273
(Pole mass: $z_0 \equiv (M - i\Gamma/2)^2$)

⋆ The analytic structure of the S matrix

$$F(s) \equiv \frac{1}{2i\rho}(S(s) - \frac{1}{S(s)}). \tag{10}$$

$$\begin{aligned} \sin(2\delta_\pi) &= \rho F \ , \\ F(s) &= \alpha + \sum_i \frac{\beta_i}{2i\rho(s_i)(s-s_i)} - \sum_j \frac{1/2i\rho(z_j^{II})}{S'(z_j^{II})(s-z_j^{II})} \\ &+ \frac{1}{\pi}\int_L \frac{\mathrm{Im}_L F(s')}{s'-s} ds' + \frac{1}{\pi}\int_R \frac{\mathrm{Im}_R F(s')}{s'-s} ds' \end{aligned} \tag{11}$$

$L = (-\infty, 0]$, $R = [4m_K^2, \infty)$; $F = 2Re_R T(s)$. Similarly (J. He, Z. Xiao and H.Z., Phys. Lett. **B536**,59,2002; Erratum-ibid.**B549**,362, 2002.),

$$\tilde{F} \equiv \frac{1}{2}(S + \frac{1}{S}) \tag{12}$$

$$\cos(2\delta_\pi) = \tilde{F} = \tilde{\alpha} + \sum_i \frac{\beta_i}{2(s-s_i)} + \sum_j \frac{1}{2S'(z_j^{II})(s-z_j^{II})}$$
$$+\frac{1}{\pi}\int_L \frac{\mathrm{Im}_L \tilde{F}(s')}{s'-s} ds' + \frac{1}{\pi}\int_R \frac{\mathrm{Im}_R \tilde{F}(s')}{s'-s} ds', \tag{13}$$

$$\Rightarrow S(z) = \cos(2\delta_\pi) + i\sin(2\delta_\pi)$$
$$= \tilde{\alpha} + i\alpha\rho(z) + \sum_i \frac{\beta_i}{2(z-s_i)} + \sum_i \frac{\rho(z)\beta_i}{2\rho(s_i)(z-s_i)}$$
$$+\sum_j \frac{\rho(z_j^{II})-\rho(z)}{2\rho(z_j^{II})S'(z_j^{II})(z-z_j^{II})} + \frac{1}{\pi}\int_L \frac{\mathrm{Im}_L \tilde{F}}{s'-z} ds' + \frac{i\rho(z)}{\pi}\int_L \frac{\mathrm{Im}_L F}{s'-z} ds'$$
$$+\frac{1}{\pi}\int_R \frac{\mathrm{Im}_R \tilde{F}}{s'-z} ds' + \frac{i\rho(z)}{\pi}\int_R \frac{\mathrm{Im}_R F}{s'-z} ds'. \tag{14}$$

Classification of the cut.

The generalized unitarity relation ($S^+S = 1$):

$$\sin^2 2\delta_\pi + \cos^2 2\delta_\pi \equiv 1 \tag{15}$$

on the whole complex s plane!

⋆ The simplest solutions of the S matrix

Simplest solutions = Solutions without dynamical cuts.

1. A bound state pole at $s = s_0$: Scattering length $a = -\sqrt{\frac{s_0}{4-s_0}}$.

$$\begin{aligned} \mathrm{Re}_R T(s) &= -\frac{1}{4}\sqrt{s_0(4-s_0)}\frac{s}{s-s_0}\ , \\ \mathrm{Im}_R T(s) &= \rho(s)\frac{s s_0}{4(s-s_0)}\ . \end{aligned} \tag{16}$$

or the S matrix,

$$S^b(s) = \frac{1 - i\rho(s)|a|}{1 + i\rho(s)|a|}\ . \tag{17}$$

2. A virtual state pole at $s = s_0$: Scattering length $a = \sqrt{\frac{s_0}{4-s_0}}$.

$$S^v(s) = \frac{1 + i\rho(s)|a|}{1 - i\rho(s)|a|} \ . \tag{18}$$

3. A (pair of) resonance: pole location: z_0 (z_0^*). The S matrix is,

$$S^R(s) = \frac{r_0[z_0] - s + i\rho(s)s\frac{\mathrm{Im}[z_0]}{\mathrm{Re}[\sqrt{z_0(z_0-4)}]}}{r_0[z_0] - s - i\rho(s)s\frac{\mathrm{Im}[z_0]}{\mathrm{Re}[\sqrt{z_0(z_0-4)}]}} \ , \tag{19}$$

where

$$r_0[z_0] = \mathrm{Re}[z_0] + \mathrm{Im}[z_0]\frac{\mathrm{Im}[\sqrt{z_0(z_0-4)}]}{\mathrm{Re}[\sqrt{z_0(z_0-4)}]} \ . \tag{20}$$

A unique solution of the unitary S matrix with only one resonance pole and without 'dynamical cuts'!

Only when the resonance locates very far from the threshold and $\mathrm{Re}[z_0] >> \mathrm{Im}[z_0]$, and when s is close to $r_0 \simeq \mathrm{Re}[z_0]$ may we go back to the Breit–Wigner solution ($\mathrm{Im}[z_0] > 0$):

$$S(s) \simeq \frac{s - z_0}{s - z_0^*} . \tag{21}$$

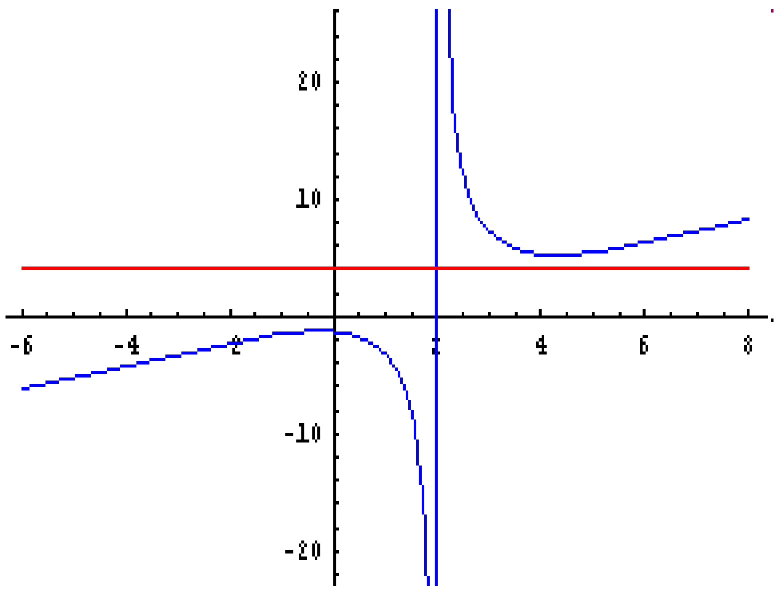

Figure 1: $r_0[z_0]$ as a function of $\mathrm{Re}[z_0]$, fixing $\mathrm{Im}[z_0] = 1$.

Phase motion:

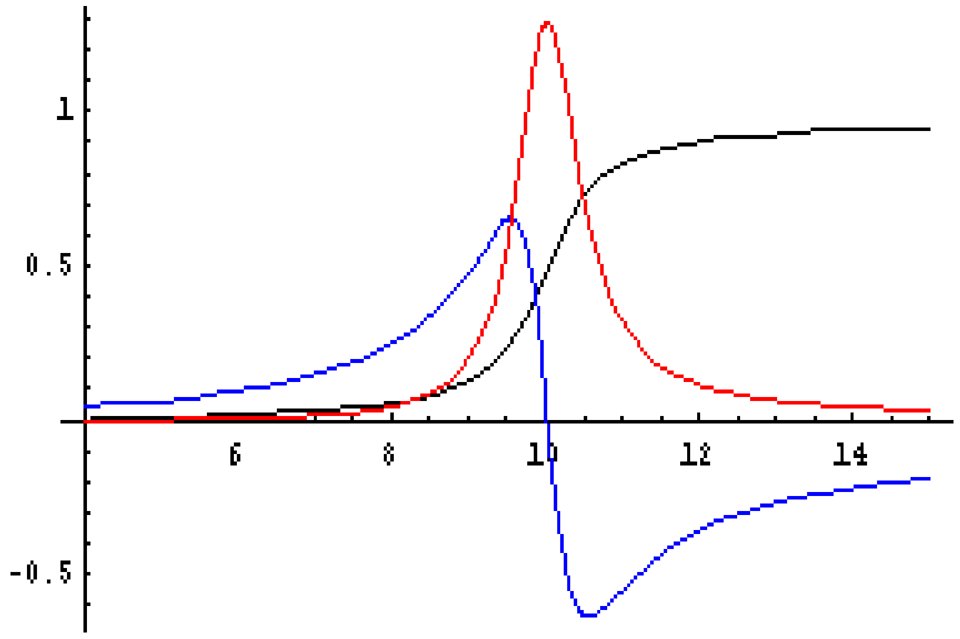

Figure 2: The phase motion of a narrow resonance. $\mathrm{Re}[z_0] = 10$, $\mathrm{Im}[z_0] = .5$.

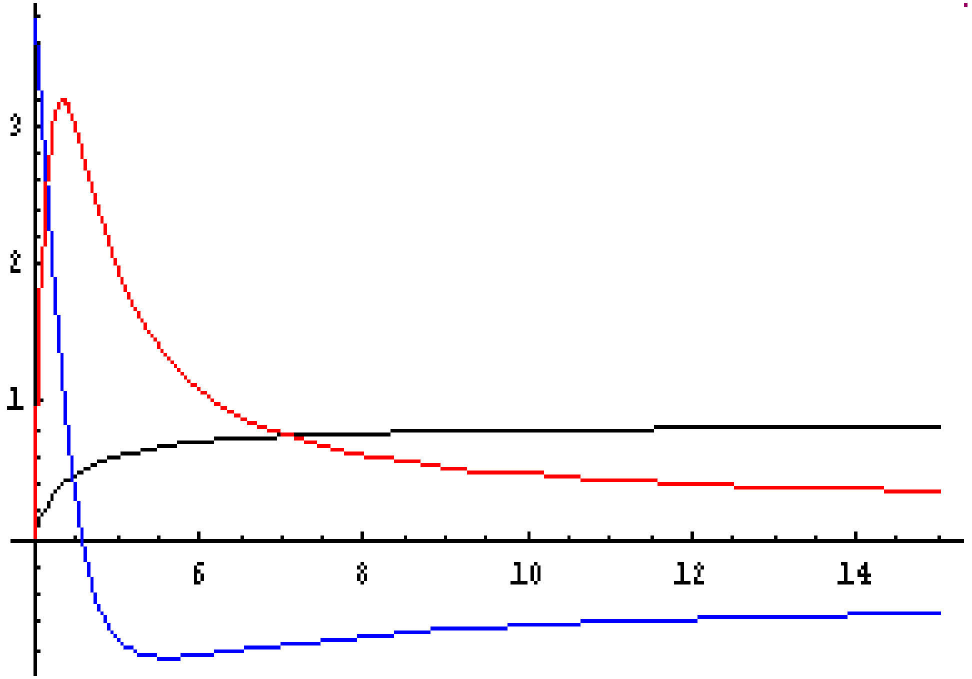

Figure 3: The phase motion of a resonance with $\mathrm{Re}[z_0] = 4$, $\mathrm{Im}[z_0] = .5$.

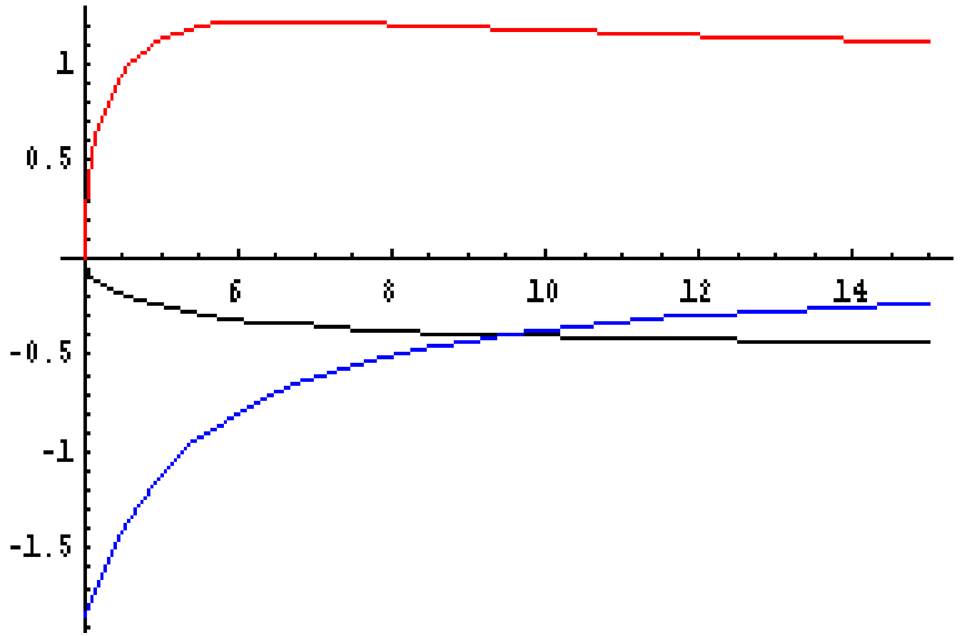

Figure 4: The phase motion of a resonance with $\mathrm{Re}[z_0] = 1.9$, $\mathrm{Im}[z_0] = .5$.

Minkowski and Ochs hep-ph/0209225

⋆ The violation of Levinson's theorem

If $r_0 > 4$, the phase pass $\pi/2$ when $s = r_0$ and

$$\delta(\infty) = \pi - \tan^{-1}\left(\frac{\mathrm{Im}[z_0]}{\mathrm{Re}[\sqrt{z_0(z_0-4)}]}\right) < \pi \ . \tag{22}$$

Actually,

$$\delta(\infty) - \delta(-\infty) = \pi \ . \tag{23}$$

⋆ The problem of a Breit–Wigner description to the σ resonance

A very common parameterization for in the literature

$$S = \frac{M^2 - s + i\rho(s)M\Gamma}{M^2 - s - i\rho(s)M\Gamma}\ , \tag{24}$$

For a sufficiently large M^2 and small $M\Gamma$ it contains a resonance and *a virtual state* (The latter is not found by χPT!) If denoting the resonance pole position as z_0, then,

$$a = \frac{\mathrm{Im}[z_0]}{\mathrm{Re}[\sqrt{z_0(z_0-4)}]}\ , \tag{25}$$

The scattering length of the two poles are additive and are both positive. Taking $M = 400$MeV, $\Gamma = 600$MeV as an example. The scattering length predicted by Eq. (19) is 0.23, and is 2.73 as predicted by Eq. (24)!

It can be dangerous to use Eq. (24)!

⋆ The the factorization of the S matrix

A unitary matrix divided by any unitary matrix is still unitary. If we single out all poles of an S matrix, we have

$$S^{phy.} = \prod_i S^{R_i} \cdot S^{cut} \ . \tag{26}$$

NO LOSS OF GENERALITY!

S^{cut}: no longer contains any pole:

$$\begin{aligned} S^{cut} &= e^{i\rho f(s)} \\ f(s) &= f_0(s) + \frac{s-4}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}_L f(s')}{(s'-4)(s'-s)} \\ &+ \frac{s-4}{\pi}\int_{4m_K^2}^{+\infty}\frac{\mathrm{Im}_R f(s')}{(s'-4)(s'-s)} \end{aligned} \tag{27}$$

Must be non-vanishing. Couple channel effects (or physics at III, IV,... sheets, etc.)

$\Rightarrow$ The phase is additive,

$$\delta(s) = \sum_i \delta_{R_i} + \delta_{b.g.} \ . \qquad (28)$$

$\delta_{b.g.}(s) = \rho(s) f(s)$.

$\star$ The background phase

A match between our parameterization and the χPT results in the physical region where χPT is reasonable.

$$\prod_i S^{r_i} \cdot S^{cut} \simeq 1 + 2i\rho(s)\mathrm{T}^{\chi PT}(s) \ , \qquad (29)$$

where

$$\mathrm{T}^{\chi PT} = T^{(2)} + T^{(4)} + T^{(6)} + \cdots \ . \qquad (30)$$

We have $f = f^{(2)} + f^{(4)} + \cdots$. Taking I=2,J=0 channel for example.

$$
\begin{aligned}
S^{cut} &= e^{2i\rho f} \simeq 1 + 2i\rho(f^{(2)} + f^{(4)}) - 2\rho^2 f^{(2)2} \\
&\simeq 1 + 2i\rho(T^{(2)} + T^{(4)}) \ ,
\end{aligned} \tag{31}
$$

$\Rightarrow$

$$
\begin{aligned}
&f = f^{(2)} + f^{(4)} + \cdots \ , \\
&f^{(2)} = T^{(2)} \ , \\
&f^{(4)} = T^{(4)} - i\rho T^{(2)2} = \mathrm{Re}_R T^{(4)} \ .
\end{aligned} \tag{32}
$$

A problem: $S(s) \sim \exp(O(-1/s)) \Rightarrow$ an essential singularity at $s = 0$! But could be acceptable except in the vicinity of $s = 0$. (Z. Xiao and H. Z., Chin. Phys. Lett. **20**(2003)342.)

This way of matching with χPT determines the background phase without introducing new parameters except those pole parameters. Avoiding the disastrous 1st poles given by Padé approximations. G. Y. Qin $et\ al.$, Phys. Lett. **B542**(2002)89.

⋆ The form-factor

is also factorizable (single channel problem).

$$A(s) = \prod_i A^{R_i}(s) \cdot A^{cut} \tag{33}$$

from Omnés solution,

$$A(s) = \exp\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta_\pi(s')}{s'(s'-s)} ds'\} \ . \tag{34}$$

Define

$$\bar{A}(s) = \frac{r_0}{r_0 - s - i\rho(s)s\frac{\mathrm{Im}[z_0]}{\mathrm{Re}[\sqrt{z_0(z_0-4)}]}} \ . \tag{35}$$

$A(s)/\bar{A}(s)$ is analytic on the right.

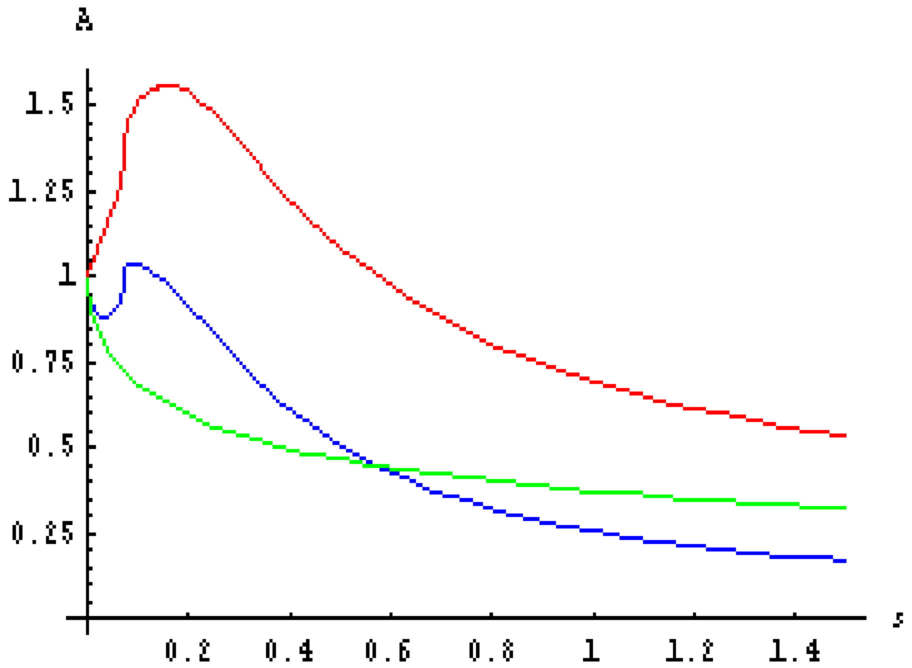

Figure 5: $|A(s)|$, the red line; $|\bar{A}(s)|$, the blue line; The green line is the ratio $|A(s)|/|\bar{A}(s)|$. For M=400MeV, $\Gamma = 600$MeV.

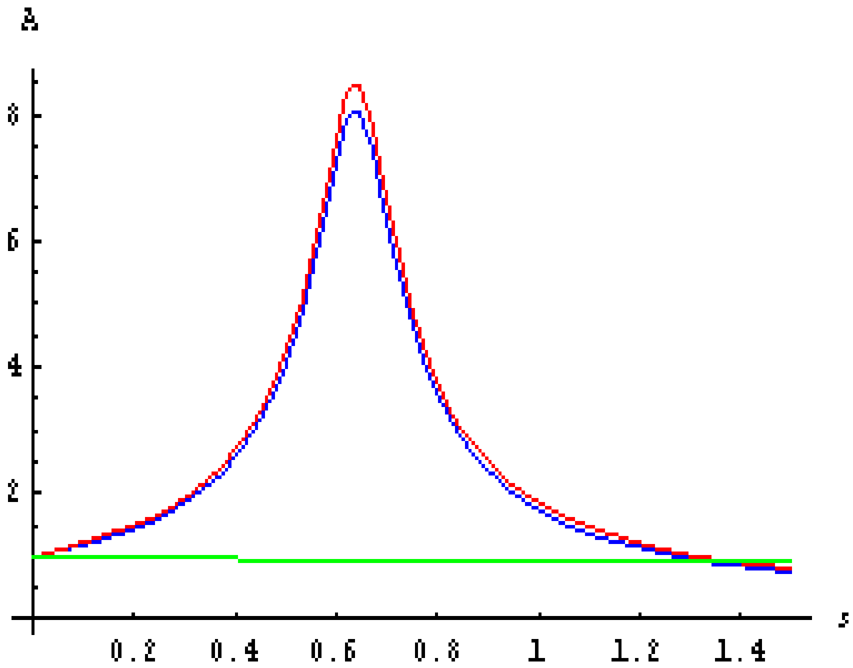

Figure 6: $|A(s)|$, the red line; $|\bar{A}(s)|$, the blue line; The green line is the ratio $|A(s)|/|\bar{A}(s)|$. For M=800MeV, $\Gamma = 100$MeV.