2001 年 11 月 7 日 (水)
日本大学素粒子論研究室コロキウム

Conformal Anomalies in Noncommutative Gauge Theories

中島　唯仁

(日大理工　一般物理)

This talk is based on a paper [hep-th/0108158]

1 Introduction

◇ Non-commutative (NC) gauge theories

- The usual product is replaced with the (Groenewold-)Moyal star product

$$f(x) * g(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu\right) f(x)g(y)\Big|_{x=y} \qquad [\, x^\mu,\, x^\nu\,]_* = i\theta^{\mu\nu}$$

Noncommutativity (Non-locality)

⇒ It might improve the renormalizablity properties of a field theory at short distances.

- UV/IR mixing [Minwalla–Raamsdonk–Seiberg]
- One-loop renormalizable [Bonora–Salizzoni]
- Unitarity [Gomis–Mehen]
- Inherent C and CP violation [Jabbari]

⇒ It smooths out any singularities in classical solutions of noncommutative field theory.

- Existence of non-singular instanton, monopole and voltex solutions [Nekrasov, Schwarz, Gross, Bak et al.]
- Nontrivial solitons $\phi * \phi = \phi$ [Gopakumar–Minwalla–Strominger]
- Topology change by gauge transformations — Solution generating technique [Harvey]

⋆ Noncommutativity brings some nontrivial consequences.

◇ Symmetries in non-commutative gauge theories

— Classical level —

- (Star-) Axial symmetry
- (Star-) Chiral gauge symmetry

  ...

— Quantum level — Anomalies occur in NC gauge theories

- Axial anomaly —— F. Ardalan & N. Sadooghi

  Naive deformation of the ordinaly axial anomalies

- Chiral gauge anomaly —— J. M. Gracia-Bondia & C. P. Martin

  More restrictive conditions for anomaly cancellation

- Anomalies and Schwinger terms —— J. Mickelsson et al.

  A "local" anomaly formula expressing the BRST cocycles

◇ Dilatation (Weyl) symmetry

— Classical level — A. Gerhold et.al

Weyl symmetry is broken in NC (massless) ϕ^4-theory

$$\delta_W S_{(*)}[\phi] = -\delta_W \theta^{\mu\nu} \frac{\partial S_{(*)}[\phi]}{\partial \theta^{\mu\nu}} \neq 0$$

How is the conformal anomaly deformed in NC gauge theories ?

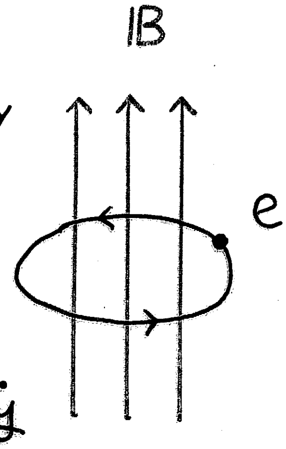
The purpose of this talk

- Evaluation of the conformal anomaly in NC gauge theory by the path integral method (Fujikawa's method).
- Calculation of β-function from the conformal anomaly in NC gauge theory.

§ 2. Noncommutative gauge theories

Background fields and noncommutativity

1. The lowest Landau level



$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + eBx\dot{y}$$

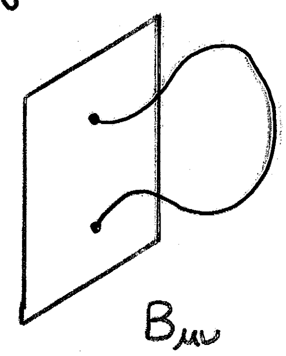
strong magnetic field $\quad \mathcal{L} \sim eBx\dot{y}$

$$\Rightarrow \quad p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} \simeq eBx \qquad \text{constraint}$$

$$\Rightarrow \quad [x, y] = -\frac{i\hbar}{eB} \neq 0 \qquad \text{noncommutative}$$

The Lagrangian is expressed with the noncommutative coordinates.

2. Strings in background B-fields (Seiberg-Witten '99)



$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma \left(g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - 2\pi i \alpha' \underbrace{B_{\mu\nu}}_{\text{constant B-field}} \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu \right)$$

Constraint (at the end points)

$$G_{\mu\nu}(\partial_\sigma X^\nu - \theta^{\nu\kappa}P_\kappa)\Big|_{\sigma = 0, \pi} = 0$$

$$G_{\mu\nu} \equiv g_{\mu\nu} - (2\pi\alpha')^2 (B g^{-1} B)_{\mu\nu}$$

$$\theta^{\mu\nu} \equiv -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{\mu\nu}$$

$$\Rightarrow [X^\mu(\tau, 0), X^\nu(\tau, 0)] = -[X^\mu(\tau, \pi), X^\nu(\tau, \pi)] = i\theta^{\mu\nu} \ (\neq 0)$$

◎ The presence of the background field leads to noncommutativity

The propagator

$$\langle X^\mu(\tau) X^\nu(\tau') \rangle$$

$$= -\alpha' G^{\mu\nu} \ln(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau')$$

The product of (tachyon) vertex operator

$$\exp(ip\cdot X)(\tau) \cdot \exp(iq\cdot X)(\tau')$$

$\alpha' \to 0$

$$\sim \exp\left(-\frac{i}{2}\theta^{\mu\nu} p_\mu q_\nu\right) \exp\left(i(p+q)\cdot X\right)(\tau')$$

$$= \exp(ip\cdot X) * \exp(iq\cdot X)$$

$$f(x) * g(x) \equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu\right) f(x) g(y)\Big|_{x=y}$$

— Moyal star product —

The effective action is expressed in terms of the open string variables $G_{\mu\nu}$ and $\theta_{\mu\nu}$.

Effective action for gauge fields

$$S = \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2} G_s} \int d^{p+1}x \sqrt{G}\, G^{\mu\kappa} G^{\nu\lambda} \hat{F}_{\mu\nu} * \hat{F}_{\kappa\lambda}$$

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i(\hat{A}_\mu * \hat{A}_\nu - \hat{A}_\nu * \hat{A}_\mu)$$

— Noncommutative gauge theory —

$\theta^{\mu\nu}$ appears only in the Moyal star product.

Moyal star product

$$[\hat{x}^1, \hat{x}^2] = i\theta \qquad (D = 2)$$

$$\hat{a} = \frac{1}{\sqrt{2\theta}} \hat{z} , \quad \hat{a}^{\dagger} = \frac{1}{\sqrt{2\theta}} \hat{\bar{z}} \qquad \hat{z} \equiv \hat{x}^1 - i\hat{x}^2$$

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \qquad \text{creation-annihilation operator}$$

Fields on the noncommutative space $\hat{\mathcal{O}}(\hat{x})$ are also operators.

Weyl ordering (one to one corresponding)

$$\hat{\mathcal{O}}[f] \equiv \int \frac{d^2k}{(2\pi)^2} \tilde{f}(k)\, e^{ik\cdot\hat{x}}$$

$$\tilde{f}(k) = \int d^2x\, f(x)\, e^{-ik\cdot x}$$

The product of two operators

$$\hat{\mathcal{O}}[f]\,\hat{\mathcal{O}}[g] \equiv \hat{\mathcal{O}}[f * g]$$

$$= \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2l}{(2\pi)^2} \tilde{f}(k)\, \tilde{g}(l) \underbrace{e^{ik\hat{x}} \cdot e^{il\hat{x}}}_{= e^{i(k+l)\hat{x}}\, e^{-\frac{i}{2} k_\mu \theta^{\mu\nu} l_\nu}}$$

$$\Longleftrightarrow \quad f * g\,(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu} \partial^x_\mu \partial^y_\nu\right) f(x)\, g(y)\Big|_{y\to x}$$

— Moyal star product —

$*$ - product is associative but noncommutative

Operator formalism $\Longleftrightarrow$ $*$ - product formalism

$\hat{\mathcal{O}}[f]\hat{\mathcal{O}}[g] \neq \hat{\mathcal{O}}[g]\hat{\mathcal{O}}[f]$ $\Longleftrightarrow$ $f * g \neq g * f$

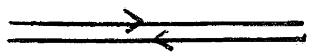
$\mathrm{Tr}_{\mathcal{H}}$ $\qquad \int dx$

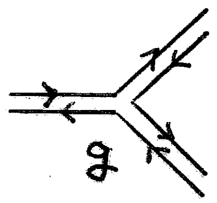
Non-planar Diagram

The product of the plane waves

$$e^{ip\cdot x} * e^{iq\cdot x} = e^{-\frac{i}{2}p\times q}\, e^{i(p+q)x}$$

Feynman rules $\qquad p\times q \equiv p_\mu \theta^{\mu\nu} q_\nu = -q\times p$

propagator $-p$  $p \qquad \dfrac{1}{p^2+m^2}$

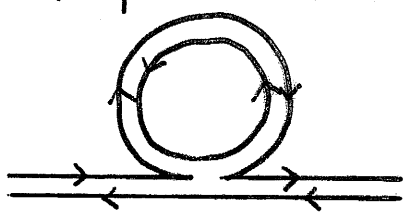
vertex p_1 

$$\frac{g}{3!}\exp\left(-\frac{i}{2}\sum p_i\times p_j\right)\delta\left(\sum p_k\right)$$

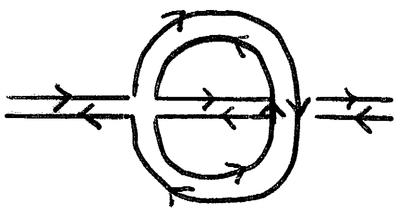
The factor is not permutation symmetric but is only cyclically symmetric

⇒ The double line notation is useful

One loop self energy diagrams



$$\sim \int \frac{d^4k}{k^2+m^2} \qquad \text{planar diagram}$$



$$\sim \int \frac{d^4k}{k^2+m^2}\, e^{ik\times p} \qquad \text{non-planar diagram}$$

⇓ cutoff

$$\sim \Lambda_{eff} - m^2 \ln\left(\frac{\Lambda_{eff}^2}{m^2}\right) + \cdots$$

$$\Lambda_{eff}^2 \equiv \frac{1}{1/\Lambda^2 + p\circ p} \qquad p\circ p \equiv p_\mu \theta^{\mu\kappa}\theta^{\nu\lambda} p_\nu g_{\kappa\lambda}$$

The non-planar diagrams are UV finite but exhibit IR singularity ⇒ UV/IR Mixing

Minwalla, Raamsdonk & Seiberg (1999)

Noncommutative QED

$$S = \int d^4x \left[-\frac{1}{4} F^{(*)}_{\mu\nu} * F^{\mu\nu}_{(*)} + i\bar{\psi} * \not{D}^{(*)}\psi - m\bar{\psi} * \psi \right]$$

$$\not{D}^{(*)}\psi = \gamma^\mu (\partial_\mu \psi - ig A_\mu * \psi)$$

$$F^{(*)}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\underbrace{(A_\mu * A_\nu - A_\nu * A_\mu)}$$

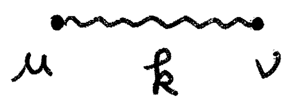
$*$ - gauge symmetry

$$\delta\psi = ig\lambda * \psi \qquad \delta A_\mu = \partial_\mu \lambda + ig[\lambda, A_\mu]_{(*)}$$

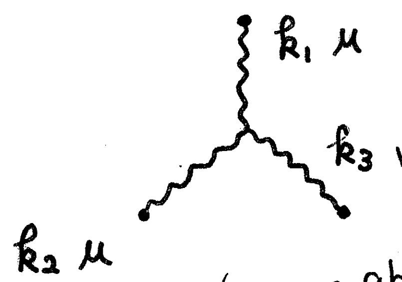
The action is invariant although the Lagrangian is not invariant.

$$\left(\mathrm{Tr} \Leftrightarrow \int d^4x \right)$$

Feynman rules



$$D_{\mu\nu}(k) = -\frac{ig_{\mu\nu}}{k^2}$$



$$W_{\lambda\mu\nu}(k_1, k_2, k_3) = -2g(2\pi)^4 \delta^4(k_1+k_2+k_3)$$
$$\times \sin\left(\frac{i}{2} k_1^\rho \theta_{\rho\alpha} k_2^\alpha\right) [g_{\lambda\mu}(k_1-k_2)_\nu + (\text{cyclic})]$$

$$\left(f^{abc} \Leftrightarrow \sin\left(\frac{i}{2} k_1^\rho \theta_{\rho a} k_2^a\right) \right)$$

β - function

$$\beta_{QED} = \frac{g^3}{12\pi^2} n_f \qquad \text{Ordinary QED}$$

$$\beta_{NC\text{-}QED} = -\frac{g^3}{(4\pi)^2}\left(\frac{22}{3} - \frac{4}{3} n_f\right) \qquad \text{NC - QED}$$

— Nonabelian - like —

Chern numbers in NC gauge theory

Axial anomaly (in ordinary gauge theory)

$$\partial_\mu J_5^\mu = \frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^a F_{\rho\sigma}{}^a \qquad SU(2)$$

⇓ Integral

$$n_+ - n_- = \frac{1}{32\pi^2} \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^a F_{\rho\sigma}{}^a \; (\in \mathbb{Z})$$

The numbers of chiral states　　　Chern numbers

— Atiyah - Singer index theorem —

Solution generating technique　　Harvey (2000)

Field strength in operator formalism

$$F = \frac{1}{\theta}([C, \bar{C}] + 1) \qquad C = a^\dagger + i\sqrt{\theta} A$$

The action

$$S = 2\pi \, \mathrm{Pf}(\theta) \, \mathrm{Tr}_{\mathcal{H}} \left(-\frac{1}{4} FF \right)$$

is invariant under the "gauge" transformation.

$$C \to C' = S^n C \bar{S}^n \qquad S = \sum |n+1\rangle\langle n| \quad \text{shift operator}$$

$$\bar{S} S = I \qquad S\bar{S} = I - |0\rangle\langle 0| \quad \text{partial isometry}$$

1st Chern number (D = 2)

$C = a^\dagger$ ($A = 0$)

$$\frac{1}{2\pi}\int F \to \theta \, \mathrm{Tr}_{\mathcal{H}} F = 0$$

$C \to C' = S^n C \bar{S}^n$

$$\frac{1}{2\pi}\int F \to \theta \, \mathrm{Tr}_{\mathcal{H}} \left(\frac{1}{\theta} \sum_{k=0}^{n-1} |k\rangle\langle k| \right) = n \; (\neq 0)$$

The integer is connected with the trace of some kind of the projection operator.

Dilatation (Weyl) symmetry in NC gauge theory

S. I. Kruglov (2001)

Dilatation (Weyl) transformation

$$\begin{cases} x^\mu \to e^{-\varepsilon} x^\mu \\ \phi(x) \to e^{\varepsilon D} \phi(e^{\varepsilon} x) \end{cases} \qquad D : [\phi] = M^D$$

$$\Rightarrow \quad j^\mu_D = x^\nu T_{\mu\nu} \qquad \text{Dilatation current}$$

$T_{\mu\nu}$ (Symmetric) Energy-momentum tensor

Dilatation (Weyl) symmetry

$$\partial_\mu j^\mu_D = 0 \quad \Leftrightarrow \quad T^\mu{}_\mu = 0$$

The energy-momentum tensor in (free) Maxwell theory

$$T_{\mu\nu} = - F_{\mu\alpha} F_\nu{}^\alpha + \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$$T^\mu{}_\mu = 0 \quad \Rightarrow \quad \text{Dilatation invariant}$$

Seiberg-Witten map up to the first order in $\theta_{\mu\nu}$

$$\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu})$$

NC Maxwell theory

$$\hat{T}^\mu{}_\mu = 2(\theta \cdot \mathbb{B})(\mathbb{E}^2 - \mathbb{B}^2) - 4(\theta \cdot \mathbb{E})(\mathbb{E} \cdot \mathbb{B}) \neq 0$$

$$\theta_i \equiv \frac{1}{2} \epsilon_{ijk} \theta_{jk} \qquad \theta_{i0} = 0$$

☆ The classical dilatation (or Conformal) invariance is broken.

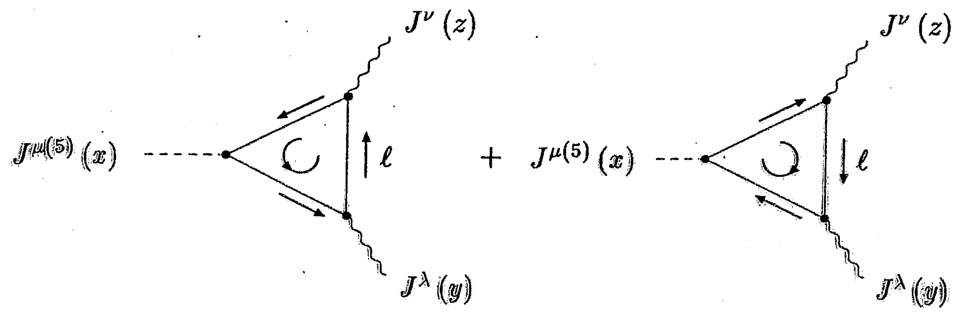
3 Anomalies in NC gauge theories

Axial anomaly in noncommutative QED

F. Ardalan & N. Sadooghi (2000)

◇ Axial current

$$< j^{\mu(5)}_{(*)}(x) >= \frac{1}{2} d^4y d^4z \Gamma^{\mu\lambda\nu}(x,y,z) * A_\lambda(y) * A_\nu(z)$$



In massless fermions, we have

$$\frac{\partial}{\partial x^\mu}\Gamma^{\mu\lambda\nu}(x,y,z)$$

$$= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{-i(p+q)\cdot x} e^{ip\cdot y} e^{iq\cdot z} [A^{\lambda\nu}(p,q,\theta) + B^{\lambda\nu}(p,q,\theta)]$$

$$A^{\lambda\nu} = -\frac{g^2}{2\pi} \cos\left(\frac{1}{2} p_\mu \theta^{\mu\nu} q_\nu\right) \varepsilon^{\lambda\nu\alpha\beta} p_\alpha q_\beta$$

$$B^{\lambda\nu} = -2ig^2 \sin\left(\frac{1}{2} p_\mu \theta^{\mu\nu} q_\nu\right)$$

$$\times \underbrace{\int \frac{d^Dl}{(2\pi)^D} \left[\mathrm{Tr}\left(\frac{1}{\not l}\gamma^\lambda\gamma^5 \frac{1}{\not l + \not q}\gamma^\nu\right) - \mathrm{Tr}\left(\frac{1}{\not l}\gamma^\lambda\gamma^5 \frac{1}{\not l - \not p}\gamma^\nu\right)\right]}_{=0}$$

$$\Rightarrow \partial_\mu < j^{\mu(5)}_{(*)}(x) >= \frac{g^2}{16\pi^2} \varepsilon^{\lambda\nu\alpha\beta} F^{(*)}_{\lambda\nu}(x) * F^{(*)}_{\alpha\beta}(x) \, .$$

- The form of the axial anomaly in NC-QED is a naive deformation of the axial anomaly in ordinary QED.
- The gauge invariance is guaranteed by the integration.

Path integral method (Fujikawa's method)

◇ Effective action

$$\exp(-\Gamma[A]) \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(-S_0^{(*)}\right) \qquad S_0^{(*)} = \int d^4x\bar{\psi} * i\not{D}^{(*)}\psi$$

$$\Rightarrow \delta^{(5)}\Gamma[A] = \int \underbrace{\mathcal{D}\tilde{\psi}\mathcal{D}\tilde{\bar{\psi}}}_{J\,\mathcal{D}\psi\mathcal{D}\bar{\psi}(J\,:\,\text{Jacobian})} \int d^4x\;\alpha(x) * \partial_\mu j^{\mu(5)}_{(*)}(x)\exp(-S_0^{(*)})$$

◇ Eigenfunctions of the Dirac operator

$$\not{D}^{(*)}\varphi_n(x) = \lambda_n\varphi_n(x)\;, \qquad \sum_n \varphi_n^\dagger(x) * \varphi_n(y) = \delta^4(x-y)$$

◇ Jacobian

$$J = \exp\left[\int d^4x\;\alpha(x) * \mathcal{A}^{(*)}(x)\right]$$

$$\begin{aligned}\Rightarrow \mathcal{A}^{(*)}(x) &= 2\sum_n \gamma_{5\,\alpha\beta}\left(\varphi_{n\beta} * \varphi^\dagger_{n\alpha}\right)(x)\\ &= 2\,\mathrm{Tr}\,\gamma_5\cdot\delta(x-x) \quad \text{(Indeterminate form)}\end{aligned}$$

The axial anomaly is derived by the following regularization,

$$\begin{aligned}\mathcal{A}^{(*)} &= \lim_{\epsilon\to 0} 2\sum_n \gamma_{5\,\alpha\beta}\;\exp_*\left(-\,\epsilon\not{D}^{(*)} * \not{D}^{(*)}\right) * \varphi_{n\beta}(x) \;*\varphi^\dagger_{n\alpha}(x)\\ &= \lim_{\epsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\Big[\,\gamma_5\\ &\qquad \exp_*\left\{-\epsilon\left((ik_\mu + D^{(*)}_\mu)^2 - \frac{i}{2}\sigma^{\mu\nu}F^{(*)}_{\mu\nu}(x)\right)\right\} * \underbrace{e^{-ik\cdot x} * e^{ik\cdot x}}_{=1}\Big]\\ &= \frac{1}{(4\pi)^2}\varepsilon^{\mu\nu\rho\sigma}F^{(*)}_{\mu\nu}(x) * F^{(*)}_{\rho\sigma}(x)\;.\end{aligned}$$

- It agrees with the result of perturbative analysis.
- The calculation is a little simple in the present method.

Chiral gauge anomaly in NC gauge theory

C. P. Martin (2000) (2001)

◇ Effective action

$$\exp\left(-\Gamma[A]\right) \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(-S_0^{(*)}\right) \qquad S_0^{(*)} = \int d^4x\bar{\psi} * i\not{D}^{(*)}P_+\psi$$

◇ Eigenfunctions of Hermitian operators

$$(i\not{D}_+^{(*)})^\dagger(i\not{D}_+^{(*)})\varphi_n = \lambda_n{}^2\varphi_n, \qquad (i\not{D}_+^{(*)})(i\not{D}_+^{(*)})^\dagger\phi_n = \lambda_n{}^2\phi_n$$

$$\sum_n \varphi_n^\dagger(x) * \varphi_n(y) = \sum_n \phi_n^\dagger(x) * \phi_n(y) = \delta^4(x-y)$$

◇ Chiral gauge anomaly (the covariant form)

$$\mathcal{A} = \lim_{\epsilon \longrightarrow 0}\int d^4x\ \alpha * \sum_n\left\{P_+\ e_*^{\left(-\ \epsilon\ (i\not{D}_+^{(*)})^\dagger(i\not{D}_+^{(*)})\right)}\varphi_n\right\} * \varphi_n^\dagger + \cdots$$

$$= \frac{1}{32\pi^2}\int d^4x\ \alpha^a *\ \mathrm{Tr}\ T_a\ \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu} * F_{\rho\sigma}$$

- The form of the chiral gauge anomalies are also a naive deformation of the ordinaly chiral gauge anomalies.
- The conditions for anomaly cancellation are restricted more due to the Moyal product.

$$\mathrm{Tr}\ T_aT_bT_c = 0 \qquad (cf\ \ \mathrm{Tr}\ T_a\{T_b, T_c\} = 0)$$

◇ Anomaly and non-planar diagram

$$D_\mu^{(*)} < j_{(*)}^{\mu\ a} >= D_\mu^{(*)}(< j_{(*)}^{(+)\mu\ a} > + < j_{(*)}^{(-)\mu\ a} >) = 0$$

$$\left(\begin{array}{l} j_{(*)}^{(+)\mu i}{}_j \equiv -i\psi^i{}_{k\ \beta} * \bar{\psi}^k{}_{j\ \alpha}(\gamma_\mu P_+)^{\alpha\beta} \\ j_{(*)}^{(-)\mu i}{}_j \equiv -i\bar{\psi}^j{}_{k\ \alpha} * \psi^k{}_{i\ \beta}(\gamma_\mu P_+)^{\alpha\beta} \end{array}\right.$$

Adjoint repr. $\simeq$ Fundamental repr. $\times$ Anti-fundamental repr.

- Theories with adjoint chiral fermion are anomaly free.

⇒ Non-planar diagrams do not contribute chiral gauge anomalies.

Chiral Gauge Theory with Adjoint Chiral Fermion

C.P. Martin hep-th/0008126

The classical action

$$S = \int d^4x \, \mathrm{Tr} \, \bar{\Psi} * (i \partial\!\!\!/ \Psi + \gamma^\mu P_+ [A_\mu \; \Psi]_*)$$

The chiral current

$$j_\mu^{a} \equiv i \frac{\delta S[A]}{\delta A^{\mu, a}} = j_\mu^{a(-)} + j_\mu^{a(+)}$$

$$\left(\begin{array}{l} j_\mu^{a(-)} = -i \, \mathrm{Tr}(\Psi_\beta * \bar{\Psi}_\alpha T_a)(\gamma^\mu P_+)_{\alpha\beta} \\ j_\mu^{a(+)} = -i \, \mathrm{Tr}(\bar{\Psi}_\alpha * \Psi_\beta T_a)(\gamma^\mu P_+)_{\alpha\beta} \end{array} \right.$$

The three-point functions

$$\Gamma_{\mu\nu\lambda}\Big|_{\text{planar}} \equiv \langle j_\mu^{(+)} j_\nu^{(+)} j_\lambda^{(+)} \rangle + \langle j_\mu^{(-)} j_\nu^{(-)} j_\lambda^{(-)} \rangle$$

$$\begin{aligned} \Gamma_{\mu\nu\lambda}\Big|_{\text{non-planar}} \equiv & \langle j_\mu^{(+)} j_\nu^{(+)} j_\lambda^{(-)} \rangle + \langle j_\mu^{(+)} j_\nu^{(-)} j_\lambda^{(+)} \rangle \\ & + \langle j_\mu^{(-)} j_\nu^{(+)} j_\lambda^{(+)} \rangle + \langle j_\mu^{(-)} j_\nu^{(-)} j_\lambda^{(+)} \rangle \\ & + \langle j_\mu^{(-)} j_\nu^{(+)} j_\lambda^{(-)} \rangle + \langle j_\mu^{(+)} j_\nu^{(-)} j_\lambda^{(-)} \rangle \end{aligned}$$

Non-planar contributions arise from fields in the adjoint representation. However, we can check that

$$p^\lambda \Gamma_{\mu\nu\lambda}(k_1, k_2 p)\Big|_{\text{planar}} = p^\lambda \Gamma_{\mu\nu\lambda}(k_1, k_2 p)\Big|_{\text{non-planar}} = 0$$

⊙ Adjoint repr. ≃ Fundamental repr. ⊗ Anti-fundamental repr.

Theories with only adjoint chiral fermions are anomaly free !!

4 Conformal anomaly and β function

◇ Dilatation (Weyl) transformation

$$\delta^{W}_{\alpha(x)} \int d^4x \mathcal{L} = \int d^4x \; \alpha(x) \; T^{\mu}{}_{\mu}(x)$$

$$T^{\mu\nu}(x) \equiv 2 \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^4x \sqrt{-g}\mathcal{L} \Big|_{g_{\mu\nu} \to \text{Minkowski metric}}$$

— Symmetric energy-momentum tensor —

$T^{\mu}{}_{\mu} = 0 \quad \Leftrightarrow \quad$ Invariant under the Weyl transformation

◇ Massless QED

$$T^{\mu}{}_{\mu}(x) \; (= m\bar{\psi}\psi) = 0$$

$\Downarrow$ [quamtum corrections] $\quad g \longrightarrow g + \alpha\beta(g)$

$$< T^{\mu}{}_{\mu}(x) > = \frac{\beta(g)}{2g^3} F_{\mu\nu}(x) F^{\mu\nu}(x) \neq 0 \qquad (\beta(g) \; : \; \beta\text{-function})$$

- Weyl symmetry is broken by the quamtum corrections.
  $\Rightarrow$ Conformal (Weyl) anomaly

- The β-function can be evaluated from the conformal anomaly.

Calculation of the conformal anomaly in ordinary QCD by the path integral method

Fujikawa (1980) (1993)

◇ Background field method

$$A_{\mu} = B_{\mu} + a_{\mu} \begin{cases} \delta B_{\mu} = 0 \; : \; \text{background field} \\ \delta a_{\mu} = D_{\mu}[B]\lambda - i[\, a_{\mu}, \lambda \,] \; : \; \text{flactuating field} \end{cases}$$

– In order to define the functional integral, we perform the gauge fixing (in the 't Hooft–Feynman gauge).

◇ Generating function at the one loop level

$$\exp\left(-\Gamma[B]\right) \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a_\mu\mathcal{D}c\mathcal{D}\bar{c}\ \exp\left(-S[\psi,\bar{\psi},a_\mu,c,\bar{c}]\right)$$

$$S = \int d^4x\ \bar{\psi}\ i\not{D}\psi + \frac{1}{2g^2}\int d^4x\ \mathrm{tr}\ \bar{c}D_\mu D^\mu c$$

$$+\frac{1}{2g^2}\int d^4x\ \mathrm{tr}\ \left\{a_\mu D_\nu D^\nu a^\mu - 2i\ \underbrace{a^\mu[\,F_{\mu\nu},\ a^\nu\,]}_{a^{\mu\,a}F_{\mu\nu}{}^c(-if^{cab})a^{\nu\,b}}\right\}$$

◇ Weyl transformations

$$a_i(x)(\equiv \sqrt[4]{|g|}e_i^\mu a_\mu(x)) \longrightarrow e^{-\alpha(x)}a_i(x)$$
$$c(x)(\equiv \sqrt[4]{|g|}c(x)) \longrightarrow e^{-2\alpha(x)}c(x)$$
$$\psi(x)(\equiv \sqrt[4]{|g|}\psi(x)) \longrightarrow e^{-\frac{1}{2}\alpha(x)}\psi(x)$$
$$\Downarrow$$
$$\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a_\mu\mathcal{D}c\mathcal{D}\bar{c}$$
$$\longrightarrow \exp\left(\int d^4x\ \alpha(x)\ \underbrace{\mathcal{A}_W}_{\text{Conformal anomaly}}\right)\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a_\mu\mathcal{D}c\mathcal{D}\bar{c}$$

◇ Conformal anomaly in ordinary QCD (SU(N) gauge group)

$$\mathcal{A}_W = n_f\cdot\mathcal{A}_W^{matter} + \mathcal{A}_W^{gauge} + \mathcal{A}_W^{ghost} \quad (n_f\ :\ \text{the number of flavour})$$
$$= \underbrace{-\frac{1}{(4\pi)^2}\left[\frac{11}{6}N - \frac{1}{3}n_f\right]}_{\beta(g)/2g^3} F_{\mu\nu}{}^a F^{\mu\nu\,a}$$

◇ β-function

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right)$$

- It is in agreement with the result of the perturbative analysis at the one loop level.

5 Conformal anomaly in NC gauge theories

— U(1) gauge theory — QED

◇ Gauge transformation $\delta a_\mu = D^{(*)}_\mu[B]\lambda - i[\, a_\mu, \lambda \,]_{(*)}$

— Naive deformation of the non-abelian gauge transformation —

◇ Effective action

$$\exp\left(-\Gamma[B] \right) \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a_\mu\mathcal{D}c\mathcal{D}\bar{c}\ \exp\left(-S_{(*)}[\psi, \bar{\psi}, a_\mu, c, \bar{c}] \right)$$

$$S^{(*)} = \int d^4x\ i\bar{\psi} * \not{D}^{(*)}\psi + \frac{1}{2g^2}\int d^4x\ \bar{c} * D^{(*)}_\mu D^\mu_{(*)}c$$

$$+\frac{1}{2g^2}\int d^4x\ \left\{ a_\mu * D^{(*)}_\nu D^\nu_{(*)}a^\mu - 2i\, a^\mu * [\, F^{(*)}_{\mu\nu},\ a^\nu \,]_{(*)} \right\}$$

Contribution from the matter fields

Weyl transformation $\psi(x) \longrightarrow e^{-\frac{1}{2}\alpha}\psi(x)$

— α : An infinitesimal constant —

◇ Eigenfunction of the Dirac operator

$$\not{D}^{(*)}\varphi_n(x) = \lambda_n\varphi_n(x)$$

◇ Jacobian $\quad J^{matter}_W = \exp\left[\, \alpha \int d^4x\ \mathcal{A}^{matter}_W \,\right]$

$$\int d^4x\ \mathcal{A}^{matter}_W \equiv \lim_{\epsilon \to 0} \sum_n \int d^4x\ \left(\exp_*(-\,\epsilon\not{D} * \not{D}) * \varphi_n(x) \right) * \varphi^\dagger_n(x)$$

$$= \lim_{\epsilon \to 0} \int d^4x \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\left(\exp_*(-\,\epsilon\not{D} * \not{D}) * e^{ik\cdot x} \right) * e^{-ik\cdot x} \right]$$

$$= \lim_{\epsilon \to 0} \int d^4x \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\exp_*\left\{ -\epsilon\left((ik_\mu + D^{(*)}_\mu)^2 - \frac{i}{2}\sigma^{\mu\nu}F^{(*)}_{\mu\nu}(x) \right) \right\} \right]$$

– The background field and its field strength do not depend on the momentum k_μ.

◇ Conformal anomaly

$$\int d^4x\ \mathcal{A}^{matter}_W(x) = \int d^4x\ \frac{1}{(4\pi)^2}\frac{2}{3}F_{\mu\nu}(x) * F^{\mu\nu}(x)$$

Contribution from the ghost fields

Weyl transformations $c(x) \longrightarrow e^{-2\alpha} c(x) \qquad \bar{c}(x) \longrightarrow \bar{c}(x)$

◇ Eigenfunction of the differential operator

$$D_\mu^{(*)} D_{(*)}^\mu S_n(x) = {\lambda'}_n^2 S_n(x)$$

The covariant derivative

$$D_\mu S_n(x) = \partial_\mu S_n(x) - i\,[B_\mu(x), S_n(x)]_* = \int \frac{d^4k}{(2\pi)^4} D_\mu[;k] \hat{S}_n(k) e^{ik\cdot x}$$

$$D_\mu[;k] \equiv \partial_\mu - i \int \frac{d^4p}{(2\pi)^4} \hat{B}_\mu(p) e^{ip\cdot x} \underbrace{(-2i)\sin\left(\frac{1}{2} p_\mu \theta^{\mu\nu} k_\nu\right)}_{\text{It corresponds to structure constants}}$$

– The background field depends on the momentum k_μ.

◇ Jacobian $\quad J_W^{ghost} = \exp\Big[\ \alpha \int d^4x\, \mathcal{A}_W^{ghost}\ \Big]$

$$\int d^4x \mathcal{A}_W^{ghost} = 2 \lim_{\epsilon \longrightarrow 0} \int d^4x \int \frac{d^4k}{(2\pi)^4} \exp_{(*)}\{-\epsilon(ik_\mu + D_\mu[;k])(ik^\mu + D^\mu[;k])\}$$

◇ The planar sector

$$\underbrace{\int d^4x\, e^{i(p+q)\cdot x}}_{(2\pi)^4 \delta(p+q)} (-2i)^2 \sin\left(\frac{1}{2} p_\mu \theta^{\mu\rho} k_\rho\right) \sin\left(\frac{1}{2} q_\nu \theta^{\nu\sigma} k_\sigma\right)$$

$$= 4 \times \left\{ \underbrace{\frac{1}{2}}_{\text{planar}} \ \underbrace{- \frac{1}{2} \cos\left(p_\mu \theta^{\mu\nu} k_\nu\right)}_{\text{non-planar}} \right\}$$

◇ Conformal anomaly

$$\int d^4x \ \mathcal{A}_W^{ghost}(x)\Big|_{planar} = \int d^4x \frac{1}{(4\pi)^2} \left(-\frac{1}{6} \times 2\right) F_{\mu\nu}(x) * F^{\mu\nu}(x)$$

Contribution from the fluctuating gauge field

Weyl transformation $\quad a_\mu(x) \longrightarrow e^{-\alpha} a_\mu(x)$

◇ Eigenfunction of the differential operator

$$D_\nu^{(*)} D^\nu_{(*)} V^\mu{}_n(x) - 2i[\, F^{\mu\nu},\ V_{\nu,n}(x)\,]_{(*)} = \lambda''{}_n V^\mu{}_n(x)$$

◇ Jacobian $\quad J_W^{gauge} = \exp\left[\, \alpha \int d^4x\, \mathcal{A}_W^{gauge} \,\right]$

$$\int d^4x \mathcal{A}_W^{gauge}$$

$$= - \lim_{\epsilon \longrightarrow 0} \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\left[\exp_{(*)}\left\{-\epsilon\left(\delta_\mu{}^\nu (ik + D[;k])^2 - 2iF_\mu{}^\nu(x;k)\right)\right\}\right]$$

$$F_{\mu\nu}(x;k) \equiv \int \frac{d^4p}{(2\pi)^4} \hat{F}_{\mu\nu}(p) e^{ip\cdot x}(-2i)\sin\left(\frac{1}{2}p_\rho \theta^{\rho\sigma} k_\sigma\right)$$

◇ Conformal anomaly

$$\int d^4x\ \ \mathcal{A}_W^{gauge}(x)\,\|_{planar} = \int d^4x\ \frac{1}{(4\pi)^2}\left(-\frac{5}{3}\times 2\right) F_{\mu\nu}(x) * F^{\mu\nu}(x)$$

Conformal anomaly in noncommutative QED

$$\int d^4x \left(n_f\ \mathcal{A}_W^{matter}(x) + \mathcal{A}_W^{gauge}(x)\|_{planar} + \mathcal{A}_W^{ghost}(x)\Big|_{planar}\right)$$

$$= \frac{1}{(4\pi)^2}\left(\frac{2}{3}n_f \times 1 - \frac{11}{6}\times 2\right) \int d^4x\, F_{\mu\nu}(x) * F^{\mu\nu}(x)$$

- The usual product $\Rightarrow$ The Moyal $*$-product
- The coefficient changes.
  - $\mathrm{tr}(T_a T_a) = 1$ —— Because of the U(1) gauge group
  - $N = 2$ —— Because of the Moyal star product

$$f^{acd} f^{bcd} = N (= C_2(G))\delta_{ab} \;\Leftrightarrow\; 4\sin^2(p_\mu \theta^{\mu\nu} q_\nu/2) = 2 + \cdots$$

$$\mathrm{tr}(color) \;\Leftrightarrow\; \int d^4x$$

β function in noncommutative QED

$$\int d^4x\, \mathcal{A}_W^{NC-QED}(x) = \frac{\beta(g)}{2g^3} \int d^4x\, F_{\mu\nu}^{(*)}(x) * F_{(*)}^{\mu\nu}(x)$$

$$\Rightarrow \beta(g)|_{NC-QED} = -\frac{g^3}{(4\pi)^2}\left(\frac{22}{3} - \frac{4}{3}n_f\right)$$

- It agrees with the result of the purturbative analysis at the one loop level. M. Hayakawa (1999), et al.

— U(N) gauge theory — QCD

It is straightforward to modify the calculation to $U(N)$ gauge group.

$$\int d^4x\; \mathcal{A}_W^{matter}(x) = \int d^4x\, \frac{1}{(4\pi)^2}\frac{2}{3}\mathrm{tr}(T_aT_b)\, F_{\mu\nu}^{(*)a}(x) * F_{(*)}^{\mu\nu b}(x)$$

$$\int d^4x\; \mathcal{A}_W^{ghost}(x)\Big|_{planar} = \int d^4x\, \frac{1}{(4\pi)^2}\left(-\frac{1}{3}\right)(f_{acd}f_{bcd})\, F_{\mu\nu}^{(*)a}(x) * F_{(*)}^{\mu\nu b}(x)$$

$$\int d^4x\; \mathcal{A}_W^{gauge}(x)\Big|_{planar} = \int d^4x\, \frac{1}{(4\pi)^2}\left(-\frac{10}{3}\right)(f_{acd}f_{bcd})\, F_{\mu\nu}^{(*)a}(x) * F_{(*)}^{\mu\nu b}(x)$$

- The trace of the product of two representation matrices occurs.

Conformal anomaly in noncommutative QCD

$$\int d^4x\, \mathcal{A}_W^{NC-QCD}(x) = \frac{1}{(4\pi)^2}\left(\frac{2}{3}n_f C(r) - \frac{11}{3}C_2(G)\right)\int d^4x\, F_{\mu\nu}^{(*)a}(x) * F_{(*)a}^{\mu\nu}(x)$$

$$\mathrm{tr}(T_aT_b) = C(r)\delta_{ab}, \quad f_{acd}f_{bcd} = C_2(G)\delta_{ab}$$

β function in noncommutative QCD

$$\beta(g)|_{NC-QCD} = -\frac{g^3}{(4\pi)^2}\left(\frac{22}{3}C_2(G) - \frac{4}{3}n_f C(r)\right)$$

This is also coincident with the β function obtained from the perturbative analysis. V. V. Khoze, G. Travaglini (2000), et al.

6 Summary

- We evaluated the conformal anomalies in noncommutative gauge theories by path integral method (Fujikawa's method)

Conformal anomaly

$$\int d^4x\, \mathcal{A}_W^{NC} = \frac{1}{(4\pi)^2}\left(\frac{2}{3}n_f C(r) - \frac{11}{3}C_2(G)\right)\int d^4x\, F_{\mu\nu}^{(*)a} * F_{(*)a}^{\mu\nu}$$

(In NC-QED, $C(r) = 1$ and $C_2(G) = 2$.)

– The local parameter $\alpha(x)$ in the Weyl transformation was made a constant parameter.

– Only the planar contribution was investigated.

– The coefficient of this conformal anomaly is different from that of the conformal anomaly in ordinary gauge theory.

β function

$$\beta(g)|_{QED} = \frac{g^3}{12\pi^2}n_f \;\Rightarrow\; \beta(g)|_{NC-QED} = -\frac{g^3}{(4\pi)^2}\left(\frac{22}{3} - \frac{4}{3}n_f\right)$$

– The β-function is modified due to the Moyal $*$-pruduct. (NC-QED)

– It is consistent with the result of the perturbative analysis.

Future problems

- Non-planar contribution to the conformal anomaly
- The conformal anomaly under the local Weyl transformation
- Extension to supersymmetry